Aggregating binary relations: voting rules and the Mean Rule via inner product

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Abstract Are plurality voting, the Kemeny rule, approval voting, the Borda count, and the Mean Rule all versions of the same aggregation rule? Yes, in a sense. The *median procedure* is a well known method (see e.g. [3], [12]) for aggregating a sequence $(R_1, R_2, \ldots, R_n)$ of binary relations, all belonging to some class of *feasible inputs*, into a single binary relation R from a possibly different class of *feasible outputs*. By relaxing the assumption that the input and output classes are related in any simple way, and expanding the range of utilized classes, we show that all the aforementioned rules are instances of the median procedure.

Most of these examples are voting rules (in which an individual input R_i can be identified with the ballot cast by voter i), but the recently introduced *Mean Rule* (see [7], [8]) is not[1]; it aggregates dichotomous weak orders D_i into a single dischotomous weak order D as outpot. One can think of this output as a collective decision as to which statements are *True* (the objects on top according to D), and which are *False* (those on bottom). The Mean Rule thus has potential spplications to *judgment aggregation*.

Our approach throughout is to view the median procedure itself as a *relational* scoring rule in which "ballots" R_i award real number scoring weights to feasible output relations R. The result of the amalgamation is then the feasible output relation accumulating the greatest total weight, as summed over all ballots. This generalizes the more traditional definition of *scoring rule*, in which ballots award scoring weights to individual alternatives (the candidates, in a voting context), rather than to binary relations on the set of alternatives (such as social rankings of candidates); the idea is thus similar to the generalizations considered in [5], [6], [16], [18], and [20]. The scoring weights are given by a Euclidean inner product $\mathcal{J}(R_i) \cdot \mathcal{J}(R)$, where $\mathcal{J}$ embeds binary relations as vertices of a hypercube in $\Re^k$. This approach yields a Euclidean form of *distance rationalization* that is universal: – the same metric is used for each instance of the median rule – as well as a *mean proximity* representation that is similarly universal.[2]

The median procedure itself is induced by the hypercube embedding $\mathcal{J}$ in particular. The potential value of alternative embeddings has not been well explored, but at least one such alternative uses a permutahedron in place of a hypercube, and its instances include some useful rules that are *not* instances of the median procedure . . . we think.

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[1] Nor is the form of cluster analysis considered in [3]; more generally, the applicability of the median procedure to amalgamation goes well beyond its application to voting.

[2] See [1], [2], [4], [9], [10], [11], [13], [14], and [17] for distance rationalization, and [20] for mean proximity representations.

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