Voting on randomly generated proposals

P. Jean–Jacques Herings and Arkadi Predtetchinski
Maastricht University

October 10, 2011

The game

In each period t:

- Nature randomly selects a policy x from a finite set X.
- The players sequentially cast a *yes* or *no* vote.
- If the proposal is approved of by a decisive set of players, the game ends, and player i receives the payoff x_i.
- If not, period $t + 1$ begins.
- In case of perpetual disagreement each player gets 0.
- Perfect information.

An example

players

alternatives

	1	2	3
a	3	0	1
b	1	3	0
c	0	1	3

The connections: bargaining

- The proposals arrive randomly.
- Perfectly patient players.

The connections: stopping games

- Solan (MoR2005): *Subgame–perfection in quitting games with perfect information and differential equations*.
- Mashiah–Yaakovi (IJGT2009): *Periodic stopping games*.
- A "collective" stopping game.

The connections: games with positive recursive payoffs

- Stochastic games where payoffs are non–negative in every absorbing state and zero in non–absorbing state.
- Flesch, Kuipers, Schoenmakers, Vrieze (MoR2008): *Subgame-Perfection in Stochastic Games with Perfect Information and Recursive Payoffs*.

The contribution

The existence of subgame perfect equilibria in pure action–independent strategies.

The plan

- Action–independence
- Back to the motivating example
- Characterizing voting equilibrium using collective response functions
- The proof

The details of the game

- Moves of nature are iid across the periods.
- The collection $\mathcal{C}(x)$ of decisive coalitions for the policy x is non–empty, monotone.
- No policy is worse than perpetual delay: $x_i \geq 0$ for each player i, policy x.

Pure strategies

A history after which player i has to move is a sequence

$$(s_0, a_0, \dots, s_{t-1}, a_{t-1}, s_t, a_t^{<i}), \text{where}$$

$s_0, \dots, s_{t-1}$	are the previously rejected proposals
$a_0, \dots, a_{t-1}$	the players' action profiles (i.e. yes–no votes),
s_t	the proposal on the table
$a_t^{<i}$	the votes of the players who vote before i.
H_i	set of histories where i moves
$\sigma_i : H_i \to \{\text{yes}, \text{no}\}$	player i's pure strategy.

Independence of actions

Definition. σ_i is *action–independent* if whenever

$$h = (s_0, a_0, \ldots, s_{t-1}, a_{t-1}, s_t, a_t^{<i})$$
$$\bar{h} = (s_0, \bar{a}_0, \ldots, s_{t-1}, \bar{a}_{t-1}, s_t, \bar{a}_t^{<i})$$

are in H_i then $\sigma_i(h) = \sigma_i(\bar{h})$.

Definition. A strategy profile $(\sigma_1, \ldots, \sigma_n)$ is a *voting equilibrium* if it is a subgame perfect equilibrium and each σ_i is a pure and action–independent strategy.

The main result

Theorem. *There exists a voting equilibrium.*

The example

players

alternatives

	1	2	3
a	3	0	1
b	1	3	0
c	0	1	3

How to play in the example

- In period 0 accept everything.
- If x has been rejected, switch to the punishment f_x.

  f_a: accept only b in stage 1 and only b and c as of stage 2.
  f_b: accept only c in stage 1 and only a and c as of stage 2.
  f_c: accept only a in stage 1 and only a and b as of stage 2.
- If f_x requires to accept y but y is rejected, switch to the punishment f_y.

Collective response function

A profile of pure action–independent strategies σ naturally induces the *collective response function* $f = f_\sigma : Seq \to \{0, 1\}$. Set $f(s_0, \ldots, s_{t-1}, s_t)$ to be 1 (0) if following the sequence $s_0, \ldots, s_{t-1}$ of rejected proposals, the profile σ dictates that s_t be accepted (rejected).

Let $f[s_0, \ldots, s_t]$ be the collective response function in a subgame following the rejection of the proposals $s_0, \ldots, s_t$.

Collective response function

Lemma. *Let $f : Seq \to \{0,1\}$ be given. There exists a voting equilibrium σ with $f = f_\sigma$ if and only if for each sequence $s = (s_0, \ldots, s_t)$*

$$\begin{array}{rcl} f(s) & = & 1 \quad \text{if} \quad s_t \in SD(U(f[s])), \\ f(s) & = & 0 \quad \text{if} \quad s_t \notin WD(U(f[s])). \end{array}$$

$$\begin{array}{rcl} SD(u) & = & \{x \in X : \{i \in N : x_i > u_i\} \in \mathcal{C}(x)\} \\ WD(u) & = & \{x \in X : \{i \in N : x_i \geq u_i\} \in \mathcal{C}(x)\}. \end{array}$$

The one–deviation principle

Lemma. *The one–deviation principle holds for the game* Γ.

The proof

I Iterative elimination of unacceptable alternatives.

II Constructing an equilibrium.

Iterative elimination of unacceptable alternatives

$$
\begin{aligned}
X_0 &= X \\
F_0 &= \{f \in F \mid f(s) = 1 \text{ if } s = (s_0, \ldots, s_t) \text{ and } s_t \in SD(U(f[s]))\}.
\end{aligned}
$$

$$
\begin{aligned}
X_1 &= \{x \in X_0 \mid \text{there is an } f \in F_0 \text{ such that } x \in WD(U(f))\} \\
F_1 &= \{f \in F_0 \mid \text{if } f(s_0, \ldots, s_t) = 1 \text{ then } s_t \in X_1\}.
\end{aligned}
$$

Iterative elimination of unacceptable alternatives

$$\begin{aligned} X_k &= \{x \in X_0 \mid \text{there is an } f \in F_{k-1} \text{ such that } x \in WD(U(f))\} \\ F_k &= \{f \in F_0 \mid \text{if } f(s_0, \ldots, s_t) = 1 \text{ then } s_t \in X_k\}. \end{aligned}$$

Lemma. *For each k the set X_k is non–empty and the function f_k is an element of F_k.*

$$f_k(s_0, \ldots, s_t) = \begin{cases} 1 & \text{if } s_t \in X_k \\ 0 & \text{otherwise.} \end{cases}$$

Iterative elimination of unacceptable alternatives

$$
\begin{aligned}
X_* &= \{x \in X_0 \mid \text{there exists an } f \in F_* \text{ such that } x \in WD(U(f))\} \\
F_* &= \{f \in F_0 \mid \text{if } f(s_0, \ldots, s_t) = 1 \text{ then } s_t \in X_*\}.
\end{aligned}
$$

Constructing an equilibrium

- For each $x \in X_*$ choose $f_x \in F_*$ with $x \in WD(U(f_x))$.
- Start with an arbitrary f_x and follow it through.
- If f_x requires to accept y but y is rejected, switch to the punishment f_y.
- Instead of starting with an f_x, we can also start with f_*. The function f_* is stationary and requires the acceptance of all policies in X_* and the rejection of all policies outside of X_*.

Constructing an equilibrium

$X_* =$ alternatives that can be accepted in a voting equilibrium

Future research

- Computing X_*.
- Extending the result to bargaining games.