A strategic approach to estate division problems with non-homogenous preferences

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Abstract

The classical bankruptcy problem (O'Neill, 1982) is extended by assuming that the agents have non-homogenous preferences over several estates. In the finite estate problem there are finitely many estates and the agents have homogenous preferences, i.e., constant utilities, per estate. In the infinite estate problem, players have arbitrary preferences over an interval of real numbers each of which is regarded as an estate. A strategic game is formulated in which each agent/player distributes his legal entitlement over the estates, resulting in individual claims per estate: each estate is then divided proportionally according to these individual claims. The focus of the paper is on the study of Nash equilibria, in particular on their existence, in finite and infinite estate games. It is also shown that, generally speaking, Nash equilibria are not unique nor Pareto optimal but that they are Pareto optimal in a second best sense: they do not Pareto dominate each other. The paper concludes with a brief consideration of envy-freeness and of two-person stability.

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1 Introduction

The classical bankruptcy problem (O'Neill, 1982; Aumann and Maschler, 1985) is the problem of dividing an estate – for instance a heritage if someone has died, or the leftovers from a bankrupt firm – among a group of claimants who have legal entitlements to the estate – for instance family members, debt holders, share holders. This problem has been approached in a normative, axiomatic

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way: see Thomson (2003) for an overview of such approaches. Starting with O'Neill (1982) and extended by Atlamaz et al. (2011), the problem has also been approached strategically, by formulating a non-cooperative game associated with the bankruptcy problem and investigating its Nash equilibria.

In the original bankruptcy problem preferences are assumed to be homogenous: basically, the estate is an amount of money and each agent wants to maximize his allocation. In the present paper preferences can be non-homogenous. We model this by assuming that there is a finite or infinite number of estates: per estate each agent's preference is homogenous, expressed by a constant utility value, but across estates an agent's preference may vary. In the case of infinitely many estates, we assume that each estate is a point in the interval $[0, 1]$, and an agent's preference is represented by a utility function defined on this interval. In a typical example, the estate is a (one-dimensional) piece of land, and agents do not only care about the amount of land they obtain but also about the quality, where different agents may value pieces of land differently. The non-cooperative game we consider is similar to the one already proposed by O'Neill (1982) and Atlamaz et al. (2011). Each agent distributes his legal entitlement over the estates, and each estate is divided proportionally to the individual claims on it. The game bears resemblance to certain procedures proposed in the fair division literature, see for instance Brams and Taylor (1996).

Throughout the paper we use the estate division terminology, but the model has applications other than the division of an heritage or the leftovers of a bankrupt firm. For instance, one may think of the interval $[0, 1]$ as a continuum of uniformly distributed consumers (cf. Hotelling, 1929), and of the claimants as firms who provide services to these consumers, with each firm providing total service equal to its entitlement. In this case, each part of the consumer continuum is distributed proportionally with respect to the claims – now called investments – of the firms. Note that this interpretation naturally allows for competitive investments by different firms in one and the same consumer segment, thus, for multiple claims in our model.

Other applications concern political problems, auctions, or problems of land division. The shares of the players may be interpreted as probabilities of winning in political elections (Merolla, Munger, and Tofias, 2005) or auctions (Cramton, Gibbons, and Klemperer, 1987). Land division has already been mentioned above; see Berliant (1985) and Berliant et al. (1992), among others. The model with finitely many estates may also be viewed as a market in which the agents have monetary endowments, rather than endowments of goods (estates): see our remarks in Section 5.

In the model with finitely many estates another potential application may arise by regarding the estates as finitely many roads from A to B. The agents are transportation companies who distribute their trucks over different roads, and the road capacity is shared proportionally to the numbers of trucks: in this application, though, it would be more realistic to let the utility of an estate, viz. the use of a road, depend on the total claim put on it. See Gairing et al. (2010).

The main part of this paper is devoted to characterizing and establishing existence of Nash equilibria. We first prove (Section 2) the existence of so-called ε-equilibria in the model with finitely many estates by a fixed point argument. In an ε-equilibrium each player is obliged to put a claim of minimally ε on each estate. Existence of Nash equilibria (i.e., $\varepsilon = 0$) is then obtained by a limit argument. A direct proof of this would be complicated by the fact that best replies need not exist. Next (Section 3) we introduce the infinite estate model and show existence of Nash equilibrium by a limit argument using the finite estate game. In Section 4 we present an example in which Nash equilibrium does not have to be unique, and we identify a special class of estate games that do have a unique Nash equilibrium. We also show that, although Nash equilibria are not Pareto optimal in general, they cannot Pareto dominate each other. We include a brief consideration of envy-freeness of equilibrium allocations, and introduce a condition called two-person stability. An allocation is two-person stable if no two players would like to swap their shares; we show that Nash equilibrium allocations possess this property. Section 5 concludes with indicating some further lines of research.

2 The finite estate problem

2.1 The finite estate problem and the finite estate game

A set $M = \{1, \ldots, m\}$ of *estates* is to be divided among a set $N = \{1, \ldots, n\}$ of *agents*. Throughout, $m \geqslant 1$ and $n \geqslant 2$. Each agent i has an *entitlement* $c_i > 0$, and a utility $u_{is} > 0$ for ownership of the complete estate s. We write $u_i = (u_{i1}, \ldots, u_{im})$. The quadruple

$$(M, N, c, u) \ ,$$

where $u = (u_1, \ldots, u_n)$ and $c = (c_1, \ldots, c_n)$, is a *finite estate problem*.

A *solution* to the finite estate problem is an allocation of shares of the estates to the agents. More precisely, if $x_{is} \geqslant 0$ is the share of estate s allocated to agent i, then $x = ((x_{is})_{s \in M})_{i \in N}$ is a solution if $\sum_{i \in N} x_{is} \leqslant 1$ for all $s \in M$. Agent i's utility from x is linear and given by $u_i(x) = \sum_{s \in M} x_{is} u_{is}$.

Remark 2.1 A few comments on the model so far are in order.

(1) In the classical bankruptcy problem of O'Neill (1982) there is only one estate ($m = 1$), and one can simply take $u_{i1} = 1$ for all $i \in N$; the entitlements c_i are monetary amounts. In our extended model the entitlements can have a more general interpretation, reflecting rights or possibilities of agents: financial slack, numbers of votes, etc. For the purpose of the present paper the entitlements could actually be normalized, e.g., by letting their sum be equal to one.

(2) The linearity of utilities allows for an interpretation of the x_{is} as probabilities: then, utility is expected utility. In turn, this enables application to indivisible estates.

The purpose of this paper is to find solutions as equilibrium outcomes of a non-cooperative game which, naturally, takes the data (c, u) of the problem into consideration. To this end, with each finite estate problem (M, N, c, u) a *finite estate game* (S, π) is associated in the following way. The *players* of the game are the agents of the estate problem. A *strategy* of player i is a vector $\sigma_i = (\sigma_{is})_{s\in M}$ of non-negative numbers, where σ_{is} denotes the claim that player i places on estate s, such that

$$\sum_{s\in M} \sigma_{is} = mc_i \ . \tag{1}$$

Remark 2.2 To interpret condition (1) one can think of the m estates as pieces of land of equal area, with total area equal to 1. The number σ_{is} is thought of as the height of player i's claim on estate s. Since the area of estate s is $1/m$, player i spends σ_{is}/m of his entitlement c_i on estate s. This interpretation and the associated condition (1) are convenient when later in the paper the finite model is extended to an infinite model of estate division. (For the finite estate model one could simply replace mc_i by c_i in (1).)

The strategy set of player i is denoted by S_i, and $S = \prod_{j\in N} S_j$ is the set of *strategy profiles*. The payoff player i receives from estate s at the strategy profile $\sigma = (\sigma_j)_{j\in N} \in S$ is

$$\pi_{is}(\sigma) = \begin{cases} \frac{\sigma_{is}}{\sum_{j=1}^{n} \sigma_{js}} \cdot u_{is} & \text{if } \sigma_{is} > 0 \\ 0 & \text{if } \sigma_{is} = 0 \ . \end{cases}$$

The total payoff of player i at the strategy profile σ is $\pi_i(\sigma) = \sum_{s\in M} \pi_{is}(\sigma)$. Clearly, $((x_{is})_{s\in M})_{i\in N}$ with $x_{is} = \sigma_{is}/\sum_{j\in N} \sigma_{js}$ if $\sigma_{is} > 0$ and $x_{is} = 0$ otherwise for all $s \in M$ and $i \in N$, is a solution to the finite estate problem: in this solution, each estate is divided among the players in proportion to their claims on it.

Definition 2.3 A strategy profile σ is a *Nash equilibrium* in (S, π) if for all $i \in N$

$$\pi_i(\sigma) \geq \pi_i(\sigma|\tau_i)$$

for all $\tau_i \in S_i$, where $(\sigma|\tau_i)$ is the strategy profile in which player i plays strategy τ_i while all other players j play σ_j.

A pictorial representation of a finite estate problem and game is given in Figure 1.

Notation For a finite estate problem (M, N, c, u) we denote $C = \sum_{i\in N} c_i$, $\underline{c} = \min\{c_i \mid i \in N\}$, $\bar{c} = \max\{c_i \mid i \in N\}$, $\underline{u} = \min\{u_{is} \mid i \in N, s \in M\}$, and $\bar{u} = \max\{u_{is} \mid i \in N, s \in M\}$.

2.2 Existence of Nash equilibrium in the finite estate game

Throughout the remainder of this section we consider the finite estate game (S, π) associated with the finite estate problem (M, N, c, u). The proof of existence of Nash equilibrium in this game is complicated by the fact that best

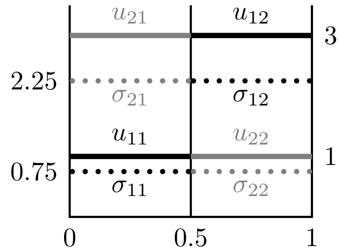

Figure 1. Pictorial representation of a finite estate division problem with $N = \{1, 2\}$, $M = \{1, 2\}$, $c_1 = c_2 = 1.5$, $u_{11} = u_{22} = 1$, $u_{12} = u_{21} = 3$. The two estates are represented by the intervals $[0, 0.5]$ and $[0.5, 1]$. The utilities of player 1 are in black, those of player 2 in gray. The following strategies are depicted: $\sigma_1 = (0.75, 2.25)$ (black dots) and $\sigma_2 = (2.25, 0.75)$ (gray dots). This strategy combination is the unique Nash equilibrium of this game, as follows from Example 4.1. The associated payoffs are 2.5 for each.

replies do not always exist. If for some estate s and strategy profile $\sigma \in S$ we have $\sigma_{js} = 0$ for all $j \neq i$, then player i does not have a best reply to σ: for τ_i to be a best reply we need $\tau_{is} > 0$ but then player i improves his payoff by decreasing τ_{is} (he still obtains all of s since no other player claims it) and increasing his claim on some estate claimed by at least one other player.

In order to circumvent this difficulty we introduce ε-equilibria and show existence of Nash equilibrium by letting ε go to zero. For any $0 \leqslant \varepsilon < \underline{c}$ let $S_i^\varepsilon = \{\sigma_i \in S_i \mid \sigma_{is} \geqslant \varepsilon \text{ for all } s \in M\}$, and $S^\varepsilon = \prod_{i \in N} S_i^\varepsilon$. Then $\sigma \in S^\varepsilon$ is an *ε-equilibrium* in (S, π) if $\pi_i(\sigma) \geqslant \pi_i(\sigma|\tau_i)$ for all $i \in N$ and $\tau_i \in S_i^\varepsilon$. Clearly, for $\varepsilon = 0$ an ε-equilibrium is a Nash equilibrium.

For a strategy profile σ, write $T_s = \sum_{i \in N} \sigma_{is}$ and $y_{is} = \sum_{j \in N \setminus \{i\}} \sigma_{js} = T_s - \sigma_{is}$. (If no confusion is likely the dependence of these numbers on σ is suppressed from notation.) Note that $\sum_{s \in M} T_s = m \sum_{i \in N} c_i = mC$. The following lemma characterizes ε-equilibria and, for $\varepsilon = 0$, Nash equilibria in (S, π). A proof can be found in the Appendix.

Lemma 2.4 *Let $0 \leqslant \varepsilon < \underline{c}$. The strategy profile $\sigma \in S^\varepsilon$ is an ε-equilibrium if and only if $T_s > 0$ for all $s \in M$ and for every player i there exists $\lambda_i > 0$ such that*

$$\lambda_i \geqslant \frac{y_{is}}{T_s^2} \cdot u_{is}$$

for all $s \in M$, with equality if $\sigma_{is} > \varepsilon$.

For each player i the number λ_i in Lemma 2.4 is the marginal utility of each positively claimed estate s in the strategy profile σ. Usually we will refer to the numbers λ_i as the marginal utilities associated with the equilibrium profile σ. A straightforward consequence of Lemma 2.4 is the following.

Lemma 2.5 *Let $0 \leqslant \varepsilon < \underline{c}$. If $\sigma \in S^\varepsilon$ is an ε-equilibrium with associated marginal utilities $\lambda_1, \ldots, \lambda_n$, then for all $i \in N$ and $s \in M$ with $\sigma_{is} > \varepsilon$ we have $\sigma_{is} = T_s - \lambda_i T_s^2 / u_{is}$.*

Our first result is that ε-equilibria exist for positive ε.

Proposition 2.6 *Let $0 < \varepsilon < \underline{c}$. Then there exists an ε-equilibrium.*

Proof. We construct a map $F : S^\varepsilon \to S^\varepsilon$, as follows. Let $\sigma \in S^\varepsilon$ with associated T_s and y_{is} for $s \in M$ and $i \in N$. For each $i \in N$, define $f_i(\lambda_i) = \sum_{s \in M} \max\{\varepsilon, T_s - \lambda_i T_s^2/u_{is}\}$ for all $\lambda_i \geqslant 0$. Then $f_i(0) \geqslant \sum_{s \in M} T_s = m \sum_{j \in N} c_j > mc_i$, and f_i is strictly decreasing until it reaches its minimum $m\varepsilon < mc_i$. Let λ_i^σ denote the unique (and positive) value of λ_i for which $f_i(\lambda_i) = mc_i$, and define $F(\sigma)_{is} = \max\{\varepsilon, T_s - \lambda_i^\sigma T_s^2/u_{is}\}$ for all $s \in M$. The map F is continuous and by Brouwer's Theorem has a fixed point σ^*.

Consider player i. Then, for all $s \in M$, $\sigma^*_{is} = F(\sigma^*)_{is} = \max\{\varepsilon, T^*_s - \lambda_i^{\sigma^*}(T^*_s)^2/u_{is}\}$, where T^*_s is associated with σ^*. Hence, $\sigma^*_{is} \geqslant T^*_s - \lambda_i^{\sigma^*}(T^*_s)^2/u_{is}$, which implies $\lambda^*_i \geqslant y^*_{is} u_{is}/(T^*_s)^2$ (where y^*_{is} is associated with σ^*), with equality if $\sigma^*_{is} > \varepsilon$. Obviously, $T^*_s > 0$ for all $s \in M$. Thus, by Lemma 2.4, σ^* is an ε-equilibrium. □

The proof of Proposition 2.6 would hold for $\varepsilon = 0$ as well if one could show that $T_s > 0$ for all s also in that case. Instead, we will prove existence of a Nash equilibrium by applying a limit argument for ε approaching 0. We first establish positive lower and upper bounds on the marginal utilities and the total claims in an ε-equilibrium for all $\varepsilon \geqslant 0$. These bounds do not depend on ε or on the number of estates m. They are used in the existence proof of Nash equilibrium in the finite estate game, and also in the infinite estate game in the next section.

Lemma 2.7 *Let $0 \leqslant \varepsilon < \underline{c}$ and let $\sigma \in S^\varepsilon$ be an ε-equilibrium with associated marginal utilities $\lambda_1, \ldots, \lambda_n$ and total claims $T_1, \ldots, T_m$.*

(*a*) *For all $i \in N$: $\lambda_i \leqslant \bar{u}/\underline{c}$.*

(*b*) *For all $s \in M$: $T_s \geqslant (n-1)\underline{u}\underline{c}/n\bar{u}$.*

(*c*) *For all $s \in M$: $T_s \leqslant 2C\bar{u}/\underline{u}$.*

(*d*) *For all $i \in N$: $\lambda_i \geqslant \underline{u}^2\underline{c}/2\bar{u}C^2$.*

Proof.
(a) Let $i \in N$. There must be an $\hat{s} \in M$ such that $\sigma_{i\hat{s}} \geqslant c_i > \varepsilon$ (since $\sum_{s \in M} \sigma_{is} = mc_i$). Then by Lemma 2.4

$$\lambda_i = \frac{y_{i\hat{s}} u_{i\hat{s}}}{T_{\hat{s}}^2} = \frac{y_{i\hat{s}}}{T_{\hat{s}}} \cdot \frac{1}{T_{\hat{s}}} \cdot u_{i\hat{s}} \leqslant 1 \cdot \frac{1}{\sigma_{i\hat{s}}} \cdot \bar{u} \leqslant \frac{\bar{u}}{c_i} \leqslant \frac{\bar{u}}{\underline{c}} .$$

(b) Let $s \in M$. Then by Lemma 2.4

$$\sum_{i \in N} \lambda_i \geqslant \sum_{i \in N} \frac{y_{is} u_{is}}{(T_s)^2} \geqslant \frac{\underline{u} \sum_{i \in N} y_{is}}{(T_s)^2} = \frac{\underline{u}(n-1)T_s}{(T_s)^2} = \frac{\underline{u}(n-1)}{T_s}. \tag{2}$$

Hence $T_s \geqslant \underline{u}(n-1)/\sum_{i\in N}\lambda_i \geqslant (n-1)\underline{u}\underline{c}/n\bar{u}$, where the last inequality follows from part (a).

(c) First, let $r \in M$ and $i,j \in N$ arbitrary. Clearly, $y_{ir} + y_{jr} \geqslant T_r$, hence $y_{ir}/T_r \geqslant 1/2$ or $y_{jr}/T_r \geqslant 1/2$. Also, by Lemma 2.4, $\lambda_i \geqslant y_{ir}u_{ir}/T_r^2$ and $\lambda_j \geqslant y_{jr}u_{jr}/T_r^2$. Hence

$$\text{for all } r \in M \text{ and } i,j \in N\text{: } \lambda_i \geqslant \underline{u}/2T_r \text{ or } \lambda_j \geqslant \underline{u}/2T_r\ . \tag{3}$$

Now take $\hat{r} \in M$ with $T_{\hat{r}} \leqslant C$ (this is possible since $\sum_{s\in M} T_s = mC$), then by applying (3) with $r = \hat{r}$ we obtain

$$\text{for all } i,j \in N\text{: } \lambda_i \geqslant \underline{u}/2C \text{ or } \lambda_j \geqslant \underline{u}/2C\ . \tag{4}$$

Let $s \in M$. If $T_s \leqslant (n-1)\underline{c} + \bar{c}$ then $T_s \leqslant 2C\bar{u}/\underline{u}$ and we are done. Otherwise, $T_s > (n-1)\underline{c} + \bar{c}$ and so there must be at least two players, say i and j, with $\sigma_{is} > \varepsilon$ and $\sigma_{js} > \varepsilon$. Then by Lemma 2.4, $\lambda_i = y_{is}u_{is}/T_s^2 \leqslant \bar{u}/T_s$ and $\lambda_j = y_{js}u_{js}/T_s^2 \leqslant \bar{u}/T_s$. By (4) this implies $\bar{u}/T_s \geqslant \underline{u}/2C$, hence $T_s \leqslant 2C\bar{u}/\underline{u}$.

(d) Let $i \in N$. There is an estate $r \in M$ such that $y_{ir}/T_r \geqslant \underline{c}/C$ since otherwise $\sum_{s\in M} y_{is} < \sum_{s\in M} T_s \cdot \underline{c}/C$, hence $m(C - c_i) < mC \cdot \underline{c}/C$ and so $C - c_i < \underline{c}$, a contradiction. Then

$$\lambda_i \geqslant \frac{y_{ir}u_{ir}}{T_r^2} \geqslant \frac{\underline{c}}{C} \cdot \frac{u_{ir}}{T_r} \geqslant \frac{\underline{c}}{C} \cdot \frac{\underline{u}}{2C\bar{u}} \cdot \underline{u} = \frac{\underline{u}^2\underline{c}}{2\bar{u}C^2}\ .$$

Here, the first inequality follows from Lemma 2.4 and the third from part (c). □

Existence of a Nash equilibrium, i.e., of a 0-equilibrium, can now be proved by letting ε approach zero, as follows.

Theorem 2.8 *The finite estate problem (S,π) has a Nash equilibrium.*

Proof. Take a decreasing sequence $0 < \varepsilon_1, \varepsilon_2, \ldots$ converging to 0 with $\varepsilon_1 < \underline{c}$, and for each $k \in \mathbb{N}$ an ε_k-equilibrium σ^k with associated marginal utilities $(\lambda_i^k)_{i\in N}$. We may assume without loss of generality that $\sigma^k \to \sigma$ for some $\sigma \in S$, and in view of Lemma 2.7 we may assume that there are $0 < \lambda_i \in \mathbb{R}$ such that $\lambda_i^k \to \lambda_i$ for all $i \in N$. Again by Lemma 2.7, all T_s associated with σ are positive. The remainder of the proof that σ is a Nash equilibrium, is then straightforward by checking the conditions in Lemma 2.4 for σ and $\varepsilon = 0$. □

3 The infinite estate problem

3.1 The infinite estate problem and game

The estates are now identified with points in the interval $[0,1]$. Each agent $i \in N$ has entitlement $c_i > 0$ and utility function $u_i : [0,1] \to (0,\infty)$ which is piecewise continuous[1] and positively bounded, i.e., there are $\underline{u}_i, \bar{u}_i \in \mathbb{R}$ such

[1] I.e., has finitely many points of discontinuity.

that $0 < \underline{u}_i \leqslant u_i(s) \leqslant \bar{u}_i$ for all $s \in [0,1]$. The pair $(c,u) = ((c_i)_{i\in N}, (u_i)_{i\in N})$ is an (*infinite*) *estate problem*.

A *solution* to such a problem is an n-tuple $(x_1, \ldots, x_n)$, where for each $i \in N$, $x_i : [0,1] \to [0,1]$ is a measurable (with respect to the Borel-Lebesgue measure on $[0,1]$) function such that $\sum_{i\in N} x_i(s) \leq 1$ for all $s \in [0,1]$. If X is a measurable subset of $[0,1]$ then agent i derives utility $\int_X x_i(s)u_i(s)\,ds$ from X. The function x_i can be interpreted as representing shares allocated to agent i, but also as a probability density function: then $\int_X x_i(s)u_i(s)\,ds$ is the expected utility agent i receives from X.

As in the finite case, we consider solutions resulting from Nash equilibria in an associated *estate game* $(\mathcal{S}, p)$, defined as follows. The players are the agents in N, and a strategy of player i is a piecewise continuous nonnegative function σ_i on $[0,1]$ which satisfies

$$\int_0^1 \sigma_i(s)ds = c_i \ .$$

The interpretation is that for each (measurable) subset $X \subseteq [0,1]$, $\int_X \sigma_i$ is the claim put by player i on X. The strategy set of player i is denoted by $\mathcal{S}_i$, and $\mathcal{S} = \prod_{i\in N} \mathcal{S}_i$ is the set of strategy profiles. For a strategy profile $\sigma = (\sigma_j)_{j\in N} \in \mathcal{S}$ and player $i \in N$, we define

$$\bar{\sigma}_i(s) = \begin{cases} \frac{\sigma_i(s)}{\sum\limits_{j=1}^{n} \sigma_j(s)} & \text{if } \sigma_i(s) > 0 \\ 0 & \text{if } \sigma_i(s) = 0 \end{cases}$$

for all $s \in [0,1]$. Then player i's payoff from σ is

$$p_i(\sigma) = \int_0^1 \bar{\sigma}_i(s)u_i(s)ds \ .$$

Definition 3.1 A strategy profile $\sigma \in \mathcal{S}$ is a *Nash equilibrium* if for all $i \in N$

$$p_i(\sigma) \geqslant p_i(\sigma|\tau_i)$$

for all $\tau_i \in \mathcal{S}_i$, where $(\sigma|\tau_i)$ is the strategy profile in which player i plays strategy τ_i while all other players j play σ_j.

The following lemma gives necessary and sufficient conditions for a strategy profile $\sigma \in \mathcal{S}$ to be a Nash equilibrium of $(\mathcal{S}, p)$. We write $T(s) = \sum_{i\in N} \sigma_i(s)$ and $y_i(s) = T(s) - \sigma_i(s)$ for all $s \in [0,1]$. A proof can be found in the Appendix.

Lemma 3.2 *The strategy profile $\sigma \in \mathcal{S}$ is a Nash equilibrium if and only if $T(s) > 0$ for almost all $s \in [0,1]$ and for every player i there exists $\lambda_i > 0$ such that*

$$\lambda_i \geqslant \frac{y_i(s)}{T(s)^2} \cdot u_i(s)$$

for almost all $s \in [0,1]$*, with equality if* $\sigma_i(s) > 0$.[2] *In the latter case,* $\sigma_i(s) = T(s) - \lambda_i T(s)^2/u_i(s)$.

3.2 Connection with the finite estate case

Let (M, N, c, u) be a finite estate problem. With this problem an infinite estate problem is associated, as follows. For each agent $i \in N$ define the function $\hat{u}_i$ on $[0,1]$ by

$$\hat{u}_i(s) = \begin{cases} u_{ik} & \text{if } s \in [\frac{k-1}{m}, \frac{k}{m}),\ k = 1, \ldots, m \\ u_{im} & \text{if } s = 1. \end{cases}$$

Then $(c, \hat{u}) = ((c_i)_{i\in N}, (\hat{u}_i)_{i\in N})$ is an infinite estate problem. Likewise, a strategy profile $\sigma \in S$ in the finite estate game (S, π) associated with (M, N, c, u) can be identified with a strategy profile $\hat{\sigma} \in \mathcal{S}$ in the infinite estate game $(\mathcal{S}, p)$ associated with $(c, \hat{u})$ by defining for each player $i \in N$

$$\hat{\sigma}_i(s) = \begin{cases} \sigma_{ik} & \text{if } s \in [\frac{k-1}{m}, \frac{k}{m}),\ k = 1, \ldots, m \\ \sigma_{im} & \text{if } s = 1. \end{cases}$$

Then, clearly, $p_i(\hat{\sigma}) = \pi_i(\sigma)$ for every player $i \in N$. The games (S, π) and $(\mathcal{S}, p)$, however, are essentially different: strategies in $\mathcal{S}_i$ do not have to be constant on each of the m intervals in $[0,1]$ corresponding to the m estates of the finite estate game. It turns out, nevertheless, that the Nash equilibria of (S, π) and $(\mathcal{S}, p)$ basically coincide.

Theorem 3.3 *Let* (M, N, c, u) *be a finite estate problem, let* $(c, \hat{u})$ *be the associated infinite estate problem, and let* (S, π) *and* $(\mathcal{S}, p)$ *be the associated finite and infinite estate games. Then* $\tau \in \mathcal{S}$ *is a Nash equilibrium in* $(\mathcal{S}, p)$ *if and only if there is a Nash equilibrium* $\sigma \in S$ *in* (S, π) *such that* $\tau = \hat{\sigma}$ *almost everywhere.*

Proof. Let $\tau \in \mathcal{S}$ be a Nash equilibrium in $(\mathcal{S}, p)$. For every $k = 1, \ldots, m$, we have essentially a homogenous claim game on the interval $X_k = [(k-1)/m, k/m]$, as studied in Atlamaz et al. (2011), with entitlements $\int_{X_k} \tau_i$ for every $i \in N$. Clearly, the restriction of τ to X_k is a Nash equilibrium in this game. By Proposition 6.3 in Atlamaz et al. (2011) it follows that each τ_i is constant almost everywhere on X_k, with constant value say σ_{ik}. Then σ is a Nash equilibrium in (S, π) and $\tau = \hat{\sigma}$ almost everywhere.

For the converse, let $\tau \in \mathcal{S}$ such that $\tau = \hat{\sigma}$ almost everywhere for some Nash equilibrium $\sigma \in S$ in (S, π). Consider player $i \in N$ and a best reply $\upsilon_i \in \mathcal{S}_i$ against τ. For each $k = 1 \ldots, m$ consider again the homogenous claim game on X_k with entitlements $\int_{X_k} \tau_j$ for players $j \neq i$ and $\int_{X_k} \upsilon_i$ for player i. Again by Proposition 6.3 in Atlamaz et al. (2011) it follows that υ_i is constant almost everywhere on each X_k. Since σ is a Nash equilibrium in (S, π) it follows that for the constant value of υ_i on X_k we can take σ_{ik} for each $k = 1, \ldots, m$. Thus, $\hat{\sigma}_i = \tau_i$ is a best reply against τ. Since i was arbitrary, it follows that τ is a Nash equilibrium in $(\mathcal{S}, p)$. □

[2] A statement is true for *almost all* $s \in [0,1]$ if it is true on a subset of $[0,1]$ of measure 1.

3.3 Existence of Nash equilibrium in the infinite estate game

Existence of a Nash equilibrium in the infinite estate game will be established by approximating this game by finite estate games. The following lemma basically states that for each vector of marginal utilities there can be at most one Nash equilibrium and, moreover, this equilibrium depends continuously on the marginal utilities. See the Appendix for the proof.

Lemma 3.4 *Let (c, u) be an infinite estate problem with associated estate game $(\mathcal{S}, p)$, let $s \in [0, 1]$, and let $\lambda_i > 0$, $i \in N$. Then the system*

$$\lambda_i \geqslant \frac{y_i(s)}{T(s)^2} \cdot u_i(s), \ \sigma_i(s)[\lambda_i - \frac{y_i(s)}{T(s)^2} \cdot u_i(s)] = 0 \text{ for all } i \in N, \tag{5}$$

where, as before, $T(s) = \sum_{i \in N} \sigma_i(s)$ and $y_i(s) = T(s) - \sigma_i(s)$, has a unique solution $\sigma(s) = (\sigma_i(s))_{i \in N}$. Moreover, $\sigma(s)$ depends continuously on $(\lambda_i, u_i(s))_{i \in N}$.

We are now sufficiently equipped to prove the existence theorem.

Theorem 3.5 *Let (c, u) be an infinite estate problem and let $(\mathcal{S}, p)$ be the associated estate game. Then $(\mathcal{S}, p)$ has a Nash equilibrium.*

Proof. For each $k \in \mathbb{N}$ let $M^k = \{1, \ldots, k\}$. For each $s \in M^k$ and $i \in N$ let $u_{is}^k = u_i(s/k)$. Then, for each $k \in \mathbb{N}$, (N, M^k, c, u^k) is a finite estate problem. Let (S^k, π^k) be the associated finite estate game. By Theorem 2.8 this game has a Nash equilibrium σ^k with associated marginal utilities $\lambda^k = (\lambda_1^k, \ldots, \lambda_n^k)$. The finite estate game (S^k, π^k) can be seen as an infinite (step function) estate game $(\mathcal{S}^k, p^k)$ and by Theorem 3.3, this has a Nash equilibrium $\tau^k = \hat{\sigma}^k$ with the same associated marginal utilities $\lambda^k = (\lambda_1^k, \ldots, \lambda_n^k)$. By Lemma 2.7 we may assume that the marginal utilities λ_i^k converge, say $\lambda_i^k \to \lambda_i > 0$ for every $i \in N$. By Lemma 3.4 we can define $\tau_i(s) = \lim_{k \to \infty} \tau_i^k(s)$ for all $s \in [0, 1]$. Since the functions τ_i are bounded by Lemma 2.7, the dominated convergence theorem of Lebesgue implies that each τ_i is integrable and $\int_0^1 \tau_i(s) ds = \lim_{k \to \infty} \int_0^1 \tau_i^k(s) ds = c_i$. Moreover, by the last claim in Lemma 3.4, each τ_i is continuous at each point at which all utility functions are continuous. Hence, the functions τ_i are piecewise continuous and therefore $\tau \in \mathcal{S}$. Also by Lemma 2.7, $T(s) := \sum_{i \in N} \tau_i(s) > 0$ for all $s \in [0, 1]$. It is straightforward to check the remaining conditions in Lemma 3.2 for τ. Hence, τ is a Nash equilibrium in $(\mathcal{S}, p)$. □

Remark 3.6 A direct proof of Theorem 3.5, i.e., a proof not based on a limiting argument, seems to be complicated by the fact that the set of bounded piecewise continuous functions or even of continuous functions on the interval $[0, 1]$ is not compact in any useful sense. This hinders application of a fixed point theorem, such as the Schauder fixed point theorem for infinite dimensional normed vector spaces (cf. Rudin, 1991).

4 Multiplicity, welfare, and equity properties

Apart from the nontrivial issue of existence of Nash equilibria in estate games, there are additional questions of interest, to which we provide some answers here.

4.1 The number of Nash equilibria

We first work out a relatively simple example, in which we have either one or three Nash equilibria. (Figure 1 in Section 2 is based on this example.)

Example 4.1 Let $M = \{1, 2\}$, $N = \{1, 2\}$, $u_{11} = u_{22} = 1$, $u_{12} = u_{21} = x > 0$, and $c_1 = c_2 = 1/2$. Lemma 2.4 implies the equilibrium conditions

$$\frac{T_1 - \sigma_{11}}{T_1^2} = \frac{T_2 - \sigma_{12}}{T_2^2} \cdot x, \ \frac{\sigma_{11}}{T_1^2} \cdot x = \frac{\sigma_{12}}{T_2^2}, \ T_1 + T_2 = 2, \ \sigma_{11} + \sigma_{12} = 1.$$

Solving this system of four equations in the unknowns T_1, T_2, σ_{11}, and σ_{12}, results in the equation

$$(x+1)^2 T^3 - 3(x+1)^2 T^2 + 2(x^2 + 6x + 1)T - 8x = 0,$$

where $T = T_1$. Since in equilibrium both T_1 and T_2 are positive, we are looking for solutions with $0 < T < 2$. As can be expected, $T = 1$ is a solution: this results in the symmetric equilibrium with $T_1 = T_2 = 1$, namely $\sigma_1 = (1/(x+1), x/(x+1))$ and $\sigma_2 = (x/(x+1), 1/(x+1))$, as can easily be derived from the four initial conditions. We next factor out $T - 1$ and obtain:

$$\begin{aligned}(x+1)^2 T^3 - 3(x+1)^2 T^2 + 2(x^2 + 6x + 1)T - 8x = \\ = (T-1)\left((x+1)^2 T^2 - 2(x+1)^2 T + 8x\right) = 0\,.\end{aligned}$$

This results in the quadratic equation

$$T^2 - 2T + \frac{8x}{(x+1)^2} = 0$$

which has solutions

$$T = 1 \pm \sqrt{1 - \frac{8x}{(x+1)^2}}$$

provided

$$\frac{8x}{(x+1)^2} < 1 \Leftrightarrow 0 < x^2 - 6x + 1 \Leftrightarrow x \notin [3 - \sqrt{8}, 3 + \sqrt{8}]\,.$$

Hence, for $3 - \sqrt{8} \leqslant x \leqslant 3 + \sqrt{8}$ the symmetric equilibrium is also the unique equilibrium. For other values of x there are three equilibria: at $x = 3 \pm \sqrt{8}$ the equilibrium manifold splits into three branches. For instance, if $x = 6$ then $T_1 = T = 1 \pm (1/7)$ and $T_2 = 1 \mp (1/7)$. For $T_1 = 8/7$ the equilibrium is $\sigma = ((8/35, 27/35), (32/35, 3/35))$; for $T_1 = 6/7$ it is $\sigma = ((3/35, 32/35), (27/35, 8/35))$. The symmetric equilibrium is $\sigma = ((1/7, 6/7), (6/7, 1/7))$.

The next example discusses a case in which the equilibrium is simple and unique. In this example the players have identical preferences.

Example 4.2 Let $v : [0, 1] \to \mathbb{R}$ be a piecewise continuous function such that $\underline{v} \leqslant v(s) \leqslant \bar{v}$ for each $s \in [0, 1]$, where $0 < \underline{v} \leqslant \bar{v}$. Assume $\int_0^1 v(s)\,ds = 1$. Consider the infinite estate problem (c, u), where each player $i \in N$ has utility function $u_i = v$.[3] Then the associated infinite estate game $(\mathcal{S}, p)$ has a Nash equilibrium σ^*, given by $\sigma_i^*(s) = c_i v(s)$ for all $s \in [0, 1]$. This equilibrium is unique up to a set of measure zero.

To see this, let σ be a Nash equilibrium with corresponding marginal utilities λ_i, $i \in N$ (cf. Lemma 3.2). Denote $C_i = C - c_i$ for all $i \in N$. Let $i, j \in N$ and suppose without loss of generality that $\lambda_i/\lambda_j \geqslant C_i/C_j$. Hence, for all $s \in [0, 1]$ with $\sigma_i(s) > 0$ we have

$$\frac{C_j}{C_i}\frac{T(s) - \sigma_i(s)}{T(s)^2}v(s) = \frac{C_j}{C_i}\lambda_i \geqslant \lambda_j \geqslant \frac{T(s) - \sigma_j(s)}{T(s)^2}v(s)$$

hence

$$\frac{C_j}{C_i}(T(s) - \sigma_i(s)) \geqslant T(s) - \sigma_j(s)\ . \tag{6}$$

Therefore

$$C_j = \frac{C_j}{C_i} \cdot C_i = \frac{C_j}{C_i}\int_0^1 (T(s) - \sigma_i(s))\,ds \geqslant \int_0^1 (T(s) - \sigma_j(s))\,ds = C_j$$

which implies that (6) holds with equality almost everywhere. Since i and j were arbitrary, we conclude that $\sigma(s) = (\sigma_1(s), \ldots, \sigma_n(s))$ is a solution of the system

$$\frac{C_1}{C_i}(T(s) - \sigma_i(s)) = T(s) - \sigma_1(s)\ , \quad i \in N \tag{7}$$

for almost all s. System (7) is a homogenous system of linear equations and so every multiple of a solution is again a solution. Therefore, we can normalize solutions by setting $T(s) - \sigma_1(s) = 1$ (clearly, $T(s) - \sigma_1(s) \neq 0$). Then, using matrix notation, (7) can be written as a non-homogenous system

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_1(s) \\ \sigma_2(s) \\ \sigma_3(s) \\ \vdots \\ \sigma_n(s) \end{bmatrix} = \begin{bmatrix} 1 \\ C_2/C_1 \\ C_3/C_1 \\ \vdots \\ C_n/C_1 \end{bmatrix}\ . \tag{8}$$

The coefficient matrix is invertible and so the system has a unique solution. It is easy to check that $\sigma(s) = c/C_1$ is this solution. Hence, any Nash equilibrium must be proportional to c, and thus σ^* is the unique Nash equilibrium (up to a set of measure zero).

[3] Or a positive multiple of v, that is irrelevant.

A noteworthy further observation is that in this unique Nash equilibrium σ^* each player plays a max-min strategy, i.e., a strategy that maximizes the minimal payoff, where the minimum is taken over all strategy combinations of the opponents. To see this, note that for any $\sigma \in \mathcal{S}$ we have

$$\sum_{i \in N} \int_0^1 \bar{\sigma}_i(t) v(t) dt = \int_0^1 \sum_{i \in N} \bar{\sigma}_i(t) v(t) dt = 1$$

i.e., the sum of the payoffs of the players is constant and equal to 1. Suppose player 1 plays σ_1^*. Following the logic above, the other players maximize their joint payoff $\sum_{i=2}^n \int_0^1 \bar{\sigma}_i(t)v(t)dt = \int_0^1 \sum_{i=2}^n \bar{\sigma}_i(t)v(t)dt$ from the strategy profile $(\sigma_1^*, \sigma_2, \ldots, \sigma_n)$ by choosing $(\sigma_2, \ldots, \sigma_n)$ such that $\sum_{i=2}^n \sigma_i(t) = \left(\sum_{i=2}^n c_i\right) v(t)$ for each $t \in [0,1]$. This implies that σ_1^* guarantees a payoff of $\int_0^1 (c_1/C)v(t)dt$ to player 1. In the same way, each player $i \in N$ can guarantee $\int_0^1 (c_i/C)v(t)dt$ by playing σ_i^*. Since the sum of all these payoffs is equal to 1, each σ_i^* is a max-min strategy.

4.2 Welfare properties of Nash equilibria

How do Nash equilibria of the (in)finite estate game perform in terms of welfare? For the homogenous case studied in Atlamaz et al. (2011) this is not so much an issue: preferences (utilities) over the estates are homogenous, there is basically only one estate and each player is indifferent between different parts of the estates and only cares about the share of the estate that he obtains. Every Nash equilibrium is therefore Pareto optimal. In the present model, where preferences are no longer homogenous and players care not only about the quantity but also the quality of the estate, this is of course different. In general, Nash equilibria cannot be expected to be Pareto optimal (see also Example 4.5 below). Nevertheless, we will show that Nash equilibria cannot Pareto dominate each other.

Let (c, u) be an infinite estate problem with associated game $(\mathcal{S}, p)$. Consider two different Nash equilibria σ and σ', with marginal utility vectors λ and λ'. Without loss of generality we assume that the conditions in Lemma 3.2 are satisfied for all $s \in [0,1]$. Recall that $\bar{\sigma}(s)$ and $\bar{\sigma}'(s)$ denote the vectors of shares of the players at each estate $s \in [0,1]$.

Lemma 4.3 *Let $i, j \in N$ with $\lambda_i'/\lambda_i \geqslant \lambda_j'/\lambda_j$ and let $s \in [0,1]$. Suppose $\bar{\sigma}_j(s) > 0$ and $\bar{\sigma}_i'(s) > \bar{\sigma}_i(s)$. Then $\bar{\sigma}_j'(s) > \bar{\sigma}_j(s)$.*

Proof. By the conditions in Lemma 3.2 we obtain

$$\frac{\lambda_j}{u_j(s)} = \frac{T(s) - \sigma_j(s)}{T(s)^2} = \frac{1}{T(s)} \left(1 - \bar{\sigma}_j(s)\right)$$

and similarly

$$\frac{\lambda_j'}{u_j(s)} \geqslant \frac{1}{T'(s)} \left(1 - \bar{\sigma}_j'(s)\right)$$

so that

$$\frac{\lambda'_j}{\lambda_j} \geqslant \frac{T(s)}{T'(s)} \frac{\left(1-\bar{\sigma}'_j(s)\right)}{(1-\bar{\sigma}_j(s))} .$$

Similarly, using $\bar{\sigma}'_i(s) > 0$, we obtain

$$\frac{\lambda'_i}{\lambda_i} \leqslant \frac{T(s)}{T'(s)} \frac{(1-\bar{\sigma}'_i(s))}{(1-\bar{\sigma}_i(s))} .$$

Hence

$$\frac{\left(1-\bar{\sigma}'_j(s)\right)}{(1-\bar{\sigma}_j(s))} \leqslant \frac{(1-\bar{\sigma}'_i(s))}{(1-\bar{\sigma}_i(s))} < 1 ,$$

which implies $\bar{\sigma}'_j(s) > \bar{\sigma}_j(s)$. □

We will use Lemma 4.3 to prove the already announced result that one Nash equilibrium cannot Pareto dominate another one. Specifically, we have the following result.

Theorem 4.4 *Let (c, u) be an infinite estate problem with associated game $(\mathcal{S}, p)$ and let σ and σ' be Nash equilibria such that $\bar{\sigma} \neq \bar{\sigma}'$ on a subset of $[0, 1]$ of positive measure. Then there are $j, k \in N$ with $p_j(\sigma) < p_j(\sigma')$ and $p_k(\sigma) > p_k(\sigma')$.*

Proof. Let O be the set of players i with $\bar{\sigma}_i(s) = \bar{\sigma}'_i(s)$ almost everywhere and let $\bar{N} = N \setminus O$. Clearly, $\bar{N}$ contains at least two players. Take player $j \in \bar{N}$ with minimal ratio λ'_j/λ_j. For almost all $s \in [0, 1]$ we have $\sum_{i \in \bar{N}} \bar{\sigma}_i(s) = \sum_{i \in \bar{N}} \bar{\sigma}'_i(s)$; so for almost all s, if $\bar{\sigma}_j(s) \neq \bar{\sigma}'_j(s)$ then there is a player $i \in \bar{N}$ with $\bar{\sigma}_i(s) < \bar{\sigma}'_i(s)$. But then Lemma 4.3 implies that $\bar{\sigma}_j(s) < \bar{\sigma}'_j(s)$. Hence, for almost all s, $\bar{\sigma}_j(s) \leqslant \bar{\sigma}'_j(s)$, with strict inequality on a subset of $[0, 1]$ of positive measure. Thus, $p_j(\sigma) < p_j(\sigma')$.

Now let $k \in \bar{N}$ be a player with maximal ratio λ'_k/λ_k. Then player k has minimal ratio λ_k/λ'_k and by an argument analogous to the preceding one we obtain $p_k(\sigma) > p_k(\sigma')$. □

Example 4.5 For the three Nash equilibria in Example 4.1 where we took $x = 6$, the payoffs are $(5.6, 4.9)$ for the equilibrium $((3/35, 27/35), (32/35, 3/35))$; $(4.9, 5.6)$ for the equilibrium $((3/35, 32/35), (27/35, 8/35))$; and approximately $(5.29, 5.29)$ for the symmetric equilibrium $((1/7, 6/7), (6/7, 1/7))$. The payoff pair $(6, 6)$, obtained by taking $\sigma_1 = (0, 1)$ and $\sigma_2 = (1, 0)$, is Pareto optimal in the game $(\mathcal{S}, p)$, and it is not difficult to show that there is no payoff pair with sum of payoffs more than 12.

Example 4.5 illustrates the fact that generally speaking Nash equilibrium payoffs are not Pareto optimal.

4.3 Envy-freeness

Envy-freeness is an appealing criterion in order to judge whether an allocation is fair or equitable. If, however, the agents in an estate problem have different entitlements then it is difficult to compare their shares in a solution to the problem. Generally speaking, larger entitlements lead to larger shares of the estates in any reasonable solution, and so an agent with a relatively small entitlement will envy an agent with a large entitlement. Therefore, we restrict attention to problems in which agents have equal entitlements. For simplicity we only consider finite estate problems.

Let (M, N, c, u) be a finite estate problem with $c_i = c_j$ for all $i, j \in N$ and with associated estate game (S, π) and let $\sigma \in S$ such that $\sum_{i \in N} \sigma_{is} > 0$ for all $s \in M$. We say that σ is *envy-free* if for all $i, j \in N$:

$$\sum_{s \in M} \frac{\sigma_{is}}{\sum_{k \in N} \sigma_{ks}} \cdot u_{is} \geqslant \sum_{s \in M} \frac{\sigma_{js}}{\sum_{k \in N} \sigma_{ks}} \cdot u_{is}\ .$$

This means that each player i weakly prefers his own allocation to that of another player j. If the opposite strict inequality holds then we say that player i *envies* player j. The following example shows that Nash equilibria need not be envy-free.

Example 4.6 Let $N = \{1, 2, 3\}$, $M = \{1, 2\}$, $c_1 = c_2 = c_3 = 1/2$ and $u_{11} = u_{22} = 10$, $u_{12} = u_{21} = u_{31} = 1$, $u_{32} = 16/25$. Then

$$\sigma_{11} = \sigma_{22} = 1,\ \sigma_{31} = \frac{2}{3},\ \sigma_{32} = \frac{1}{3}$$

is a Nash equilibrium, and player 3 envies player 1.

The next result shows that for two players every Nash equilibrium is envy-free.

Theorem 4.7 *Let $(M, \{1, 2\}, c, u)$ be a finite estate problem with $c_1 = c_2$, and let σ be a Nash equilibrium in the associated game (S, π). Then σ is envy-free.*

Proof. We shall prove that

$$\sum_{s \in M} \frac{\sigma_{2s} - \sigma_{1s}}{T_s} \cdot u_{1s} \leqslant 0$$

which means that player 1 does not envy player 2. Since $\sigma_{is} > 0$ for $i = 1, 2$ and for all $s \in M$, we have $(y_{is}/T_s^2) \cdot u_{is} = (y_{i1}/T_1^2) \cdot u_{i1}$, hence $u_{is} = (T_s^2/y_{is})(y_{i1}/T_1^2) \cdot u_{i1}$, for $i = 1, 2$ and $s \in M$. Then

$$\sum_{s \in M} \frac{\sigma_{2s} - \sigma_{1s}}{T_s} \cdot u_{1s} = \frac{\sigma_{21} - \sigma_{11}}{T_1} \cdot u_{11} + \sum_{s \in M \setminus \{1\}} \frac{\sigma_{2s} - \sigma_{1s}}{T_s} \frac{T_s^2}{y_{1s}} \frac{y_{11}}{T_1^2} \cdot u_{11}.$$

Since we just want show that the sum is negative, we can multiply by the positive number $T_1^2/(y_{11} \cdot u_{11})$. This yields

$$\sum_{s\in M} (\sigma_{2s} - \sigma_{1s}) \frac{T_s}{y_{1s}}.$$

Now, since $y_{1s} = \sigma_{2s}$ and $\sigma_{1s} + \sigma_{2s} = T_s$, we have

$$\sum_{s\in M} (\sigma_{2s} - \sigma_{1s}) \frac{T_s}{y_{1s}} = \sum_{s\in M} (\sigma_{2s} - \sigma_{1s}) \frac{T_s}{\sigma_{2s}} = \sum_{s\in M} \frac{2\sigma_{2s} - T_s}{\sigma_{2s}} T_s = \sum_{s\in M} 2T_s - \frac{T_s^2}{\sigma_{2s}}.$$

For any given $(T_s)_{s\in M}$ the σ' that maximizes the expression on the right-hand side under the constraint $\sum_{s\in M} \sigma'_{2s} = m \cdot c_2$ satisfies the property that T_s/σ'_{2s} is constant for all s, as can easily be checked using Lagrange. Since $\sum_{s\in M} T_s / \sum_{s\in M} \sigma'_{2s} = m(c_1 + c_2)/mc_2 = 2mc_2/mc_2 = 2$, we have $T_s/\sigma'_{2s} = 2$ for all s. Hence

$$\sum_{s\in M} 2T_s - \frac{T_s^2}{\sigma_{2s}} \leqslant \sum_{s\in M} 2T_s - \frac{T_s}{\sigma'_{2s}} T_s = \sum_{s\in M} 2T_s - 2T_s = 0 .$$

Hence player 1 does not envy player 2. Similarly one shows that player 2 does not envy player 1. □

In the proof of this theorem it is used that in a Nash equilibrium every estate is claimed by at least two players, so that in a two-player estate game every player puts a positive claim on every estate in an equilibrium. If there are more than two players then even if every player puts a positive claim on every estate in a Nash equilibrium, this does not imply that such an equilibrium is envy-free, as the next example shows.

Example 4.8 Let $N = \{1, 2, 3\}$, $M = \{1, 2\}$, $c_1 = c_2 = c_3 = 6$, $u_{i1} = 1$ for $i = 1, 2, 3$, $u_{12} = 16/35$, $u_{22} = 48/25$, $u_{32} = 16/25$. Then

$$\sigma_{11} = \sigma_{22} = 5,\ \sigma_{12} = \sigma_{21} = 1, \sigma_{31} = 4,\ \sigma_{32} = 2$$

is a Nash equilibrium, and player 3 envies player 1.

4.4 Two-person stability

An interesting weakening of the envy-freeness condition is obtained by requiring that no two players would like to swap their allocations. More precisely, we say that that a strategy profile σ is *two-person stable* if for all $i, j \in N$ we have $\Delta u_i + \Delta u_j \leq 0$, where

$$\Delta u_i = \sum_{s\in M} \frac{\sigma_{js} - \sigma_{is}}{T_s} u_{is} \ \text{ and } \ \Delta u_j = \sum_{s\in M} \frac{\sigma_{is} - \sigma_{js}}{T_s} u_{js} .$$

It turns out that any Nash equilibrium profile is two-person stable.

Theorem 4.9 *Let (M, N, c, u) be a finite estate problem and let σ be a Nash equilibrium in the associated game (S, π). Then σ is two-person stable.*

Proof. Let $i, j \in N$, $i \neq j$, and let $\lambda_i, \lambda_j > 0$ be the marginal utilities associated with σ for players i and j. We will show that $\lambda_j \Delta u_i + \lambda_i \Delta u_j \leq 0$, from which the theorem follows.

First let s be an estate such that $\sigma_{is}, \sigma_{js} > 0$. Then Lemma 2.4 implies

$$\frac{y_{is}}{T_s}\lambda_j u_{is} = \frac{y_{js}}{T_s}\lambda_i u_{js} \ ,$$

and by subtracting $\frac{y_{js}}{T_s}\lambda_j u_{is}$ we obtain

$$\frac{\sigma_{js} - \sigma_{is}}{T_s}\lambda_j u_{is} = \frac{y_{is} - y_{js}}{T_s}\lambda_j u_{is} = \frac{y_{js}}{T_s}\left(\lambda_i u_{js} - \lambda_j u_{is}\right) \ . \tag{9}$$

Similarly

$$\frac{\sigma_{is} - \sigma_{js}}{T_s}\lambda_i u_{js} = -\frac{\sigma_{js} - \sigma_{is}}{T_s}\lambda_j u_{is}\frac{\lambda_i u_{js}}{\lambda_j u_{is}} = -\frac{y_{js}}{T_s}\left(\lambda_i u_{js} - \lambda_j u_{is}\right)\frac{\lambda_i u_{js}}{\lambda_j u_{is}} \ . \tag{10}$$

The sum of the right-hand sides of (9) and (10) is equal to

$$\frac{y_{js}}{T_s}\left(2\lambda_i u_{js} - \lambda_j u_{is} - \frac{\lambda_i^2}{\lambda_j}\frac{u_{js}^2}{u_{is}}\right) \ ,$$

which is nonpositive since

$$2\lambda_i\lambda_j u_{is}u_{js} - \lambda_j^2 u_{is}^2 - \lambda_i^2 u_{js}^2 = -\left(\lambda_i u_{is} + \lambda_j u_{js}\right)^2 \leq 0 \ .$$

Therefore, the sum of the left-hand sides of (9) and (10) is nonpositive, i.e.,

$$\frac{\sigma_{js} - \sigma_{is}}{T_s}\lambda_j u_{is} + \frac{\sigma_{is} - \sigma_{js}}{T_s}\lambda_i u_{js} \leq 0 \ . \tag{11}$$

We next show that (11) also holds for estates that are not positively claimed by both players. If $\sigma_{is} = \sigma_{js} = 0$ then obviously $\frac{\sigma_{js}-\sigma_{is}}{T_s}\lambda_j u_{is} = \frac{\sigma_{is}-\sigma_{js}}{T_s}\lambda_i u_{js} = 0$. Now let s be an estate with $\sigma_{is} > 0$ and $\sigma_{js} = 0$. Then by Lemma 2.4

$$\frac{\sigma_{js} - \sigma_{is}}{T_s}\lambda_j u_{is} \geq \frac{y_{js}}{T_s}\left(\lambda_i u_{js} - \lambda_j u_{is}\right)$$

so that

$$0 \geq \frac{-\sigma_{is}}{T_s}u_{is} \geq \lambda_i u_{js} - \lambda_j u_{is}$$

which implies

$$0 \geq \frac{\sigma_{is}}{T_s}\left(\lambda_i u_{js} - \lambda_j u_{is}\right) = \frac{\sigma_{is}}{T_s}\lambda_i u_{js} - \frac{\sigma_{is}}{T_s}\lambda_j u_{is} = \frac{\sigma_{is} - \sigma_{js}}{T_s}\lambda_i u_{js} + \frac{\sigma_{js} - \sigma_{is}}{T_s}\lambda_j u_{is}.$$

By summing, finally, over all estates we obtain the desired result. □

It is an open question whether a similar stability condition also holds for three or more players.

5 Concluding remarks

The analysis in Section 4 suggests a number of directions for further research. One of these is to investigate the cardinality and the stability of the set of Nash equilibria in detail. Another interesting question is how well Nash equilibria perform in terms of welfare. For instance, one can ask how large the loss of welfare is compared to a Pareto optimal division of the estates.[4]

Also the question of fairness deserves further exploration. Are there other fairness criteria, weaker than envy-freeness, that are satisfied in equilibrium, like the two-person stability condition considered in Section 4?

In a wider perspective, the mechanism design aspect is of interest. For instance, in the claim game payoff principles other than the proportional one can be studied and compared in terms of welfare and equity.

Finally, the finite estate division problem can be viewed as a market economy, with the estates interpreted as goods (one unit of each good present) and the entitlements as monetary endowments of the agent. One the one hand one may consider Walrasian equilibrium in this economy, on the other hand the associated market game as a game with transferable utility (cf. Shapley and Shubik, 1969) or as a game with non-transferable utility (cf. Shubik, 1959); and the relation with Nash equilibrium, if any. For infinite estate division problems Walrasian equilibrium is studied in Berliant (1985).

A Appendix

Proof of Lemma 2.4. For the only-if part, let $\sigma \in S^{\varepsilon}$ be an ε-equilibrium. If $\varepsilon > 0$ then $T_s \geqslant n\varepsilon > 0$ for each $s \in M$; if $\varepsilon = 0$ then $T_s = 0$ for some s would imply that each player can improve by putting a small claim on s, a contradiction. Consider player $i \in N$, then σ_i solves the problem

$$\max_{\tau_i \in S_i} \sum_{s \in M} \frac{\tau_{is} u_{is}}{y_{is} + \tau_{is}} \text{ subject to } \textstyle\sum_{s \in M} \tau_{is} = mc_i \text{ and } \tau_{is} \geqslant \varepsilon \text{ for all } s \in M .$$

Considering the Lagrangian for this problem with multiplier λ_i for the constraint $\sum_{s \in M} \tau_{is} = mc_i$, it follows from the Kuhn-Tucker conditions that

$$\lambda_i \geqslant \frac{d}{d\tau_{is}} \left(\frac{\tau_{is} u_{is}}{y_{is} + \tau_{is}} \right)_{|\tau_{is} = \sigma_{is}} = \frac{y_{is} u_{is}}{(y_{is} + \sigma_{is})^2} = \frac{y_{is} u_{is}}{T_s^2} \quad \text{for all } s \in M$$

with equality if $\sigma_{is} > \varepsilon$. It is easy to see that λ_i must be positive: otherwise, $y_{is} = 0$ for all s, which means that for all $j \neq i$ we would have $\sum_{s \in M} \sigma_{js} = 0$, a plain contradiction.

For the if-part, it is sufficient to observe that for every player i, under the conditions stated, σ_i is a stationary point of the Lagrangian with multiplier λ_i and that the Lagrangian is concave. □

[4] I.e., what is the 'price of anarchy'? Cf. Johari and Tsitsiklis, 2004.

Proof of Lemma 3.2. For the only-if part, let σ be a Nash equilibrium. If $T(s) = 0$ for all $s \in X$ where X would have measure larger than zero, then an arbitrary player could improve by putting a very small positive claim on X. Consider player i, then σ_i solves the maximization problem

$$\max_{\tau_i} \int_0^1 \frac{\tau_i(s)}{y_i(s) + \tau_i(s)} u_i(s)\, ds \quad \text{subject to} \quad \int_0^1 \tau_i(s)\, ds = c_i.$$

This maximization problem can be formulated as an optimal control problem

$$\max_x \int_0^1 \frac{x(s) u_i(s)}{y_i(s) + x(s)}\, ds$$
$$\text{subject to} \quad \dot{X}(s) = -x(s),\ x(s) \geqslant 0,\ X(0) = c_i,\ X(1) = 0$$

with piecewise continuous control variable $x(s)$. The corresponding Lagrangian is

$$L = \frac{x(s) u_i(s)}{y_i(s) + x(s)} - \pi(s) x(s) + \mu(s) x(s)$$

and the necessary conditions for an optimum are

$$\frac{\partial L}{\partial x(s)} = 0,\ \frac{\partial L}{\partial \pi(s)} = \dot{X}(s),\ \frac{\partial L}{\partial X(s)} = -\dot{\pi}(s),\ \mu(s)x(s) = 0,\ \mu(s) \geqslant 0,\ x(s) \geqslant 0.$$

Since $\partial L / \partial X(s) = 0$, there is a λ_i such that $\pi(s) = \lambda_i$ for all $s \in [0,1]$. Then, $\partial L / \partial x(s) = 0$ implies

$$\frac{y_i(s) u_i(s)}{(y_i(s) + x(s))^2} - \lambda_i + \mu(s) = 0 \quad \text{for all } s \in [0,1].$$

Hence, if $x(s) > 0$ then $\mu(s) = 0$ and $\lambda_i = \frac{y_i(s)u_i(s)}{(y_i(s)+x(s))^2}$, and if $x(s) = 0$ then $\mu(s) \geqslant 0$ and $\lambda_i \geqslant \frac{y_i(s)u_i(s)}{(y_i(s)+x(s))^2}$. Furthermore, $x(s) \geqslant 0$ for all s and $\int_0^1 x(s)\, ds = c_i$ since $\dot{X}(s) = -x(s)$ for all s. Hence, all these conditions are satisfied for σ_i. Clearly, λ_i must be positive.

For the if-part, we provide the following direct argument. Let the stated conditions hold for $\sigma \in \mathcal{S}$, let $i \in N$ and $\sigma' \in \mathcal{S}_i$. We show that player i's payoff from σ' is not larger than his payoff from σ, i.e., that

$$\int_0^1 \frac{\sigma_i'(s)}{y_i(s) + \sigma_i'(s)} \cdot u_i(s) ds \leqslant \int_0^1 \frac{\sigma_i(s)}{y_i(s) + \sigma_i(s)} \cdot u_i(s) ds.$$

Without loss of generality we have assumed here that the denominators in both integrals are positive for every $s \in [0,1]$: for the right-hand side the denominator is positive almost everywhere by assumption, and for the left-hand side we may approximate σ' arbitrarily close by a strategy of player i that is positive everywhere. Now

$$\int_0^1 \frac{\sigma_i'(s)}{y_i(s)+\sigma_i'(s)} \cdot u_i(s)ds - \int_0^1 \frac{\sigma_i(s)}{y_i(s)+\sigma_i(s)} \cdot u_i(s)ds$$

$$\begin{aligned}
&= \int_0^1 \frac{(\sigma_i'(s)-\sigma_i(s))\,y_i(s)}{(y_i(s)+\sigma_i'(s))\,(y_i(s)+\sigma_i(s))} \cdot u_i(s)ds \\
&= \int_0^1 (\sigma_i'(s)-\sigma_i(s))\,\frac{y_i(s)+\sigma_i(s)}{y_i(s)+\sigma_i'(s)} \frac{y_i(s)}{(y_i(s)+\sigma_i(s))^2} \cdot u_i(s)ds \\
&\leqslant \int_0^1 (\sigma_i'(s)-\sigma_i(s))\,\frac{y_i(s)+\sigma_i(s)}{y_i(s)+\sigma_i'(s)}\lambda_i ds.
\end{aligned}$$

For all $s \in [0,1]$ we have

$$\sigma_i'(s)-\sigma_i(s) > 0 \Longleftrightarrow \frac{y_i(s)+\sigma_i(s)}{y_i(s)+\sigma_i'(s)} < 1$$

hence

$$(\sigma_i'(s)-\sigma_i(s))\,\frac{y_i(s)+\sigma_i(s)}{y_i(s)+\sigma_i'(s)} \leqslant \sigma_i'(s)-\sigma_i(s).$$

It follows that

$$\begin{aligned}
\int_0^1 (\sigma_i'(s)-\sigma_i(s))\,\frac{y_i(s)+\sigma_i(s)}{y_i(s)+\sigma_i'(s)}\lambda_i ds &\leqslant \int_0^1 (\sigma_i'(s)-\sigma_i(s))\,\lambda_i ds \\
&= (c_i - c_i)\,\lambda_i = 0,
\end{aligned}$$

which completes the proof of the if-part.

The last claim in the lemma, $\sigma_i(s) = T(s) - \lambda_i T(s)^2/u_i(s)$ if $\sigma_i(s) > 0$, is straightforward. □

The following lemma is an auxiliary result used in the proof of Lemma 3.4 below.

Lemma A.1 *Let $(a_1, \ldots, a_n)$ be a vector of positive numbers and let $(a_{(1)}, \ldots, a_{(n)})$ denote a permutation of these numbers such that $a_{(1)} \leqslant \ldots \leqslant a_{(n)}$. Then*

(*i*) *There is a unique $j \in \{1, \ldots, n\}$ such that*

$$(j-1)a_{(j)} < \sum_{i=1}^{j} a_{(i)} \leqslant (j-1)a_{(j+1)}, \tag{12}$$

where $a_{(n+1)} = \infty$. If $j^* = j^*(a_1, \ldots, a_n)$ is this unique value, then $j^* \geqslant 2$.

(ii) *The function* $T : (a_1, \ldots, a_n) \mapsto (j^* - 1)/\sum_{i=1}^{j^*} a_{(i)}$ *with* $j^* = j^*(a_1, \ldots, a_n)$*, is continuous.*

Proof of Lemma A.1. (i) First note that for $j = 1$ the inequalities in (12) imply $0 < a_{(i)} \leqslant 0$, an impossibility. Suppose that (12) does not hold for any $j \in \{2, \ldots, n-1\}$. We claim that (12) holds for $j = n$. If not, then $(n-1)a_{(n)} \geqslant \sum_{i=1}^{n} a_{(i)}$. This implies $(n-2)a_{(n)} \geqslant \sum_{i=1}^{n-1} a_{(i)}$, hence $(n-2)a_{(n-1)} \geqslant \sum_{i=1}^{n-1} a_{(i)}$ since otherwise $j = n-1$ would satisfy (12). Repeating this argument, we find $a_{(2)} \geqslant a_{(1)} + a_{(2)}$, a contradiction. Hence, (12) holds for some $j \in \{2, \ldots, n\}$. Let $j^* \geq 2$ be the minimal j for which (12) holds. By the second inequality in (12) we have $(j^* - 1)a_{(j^*+1)} \geqslant \sum_{i=1}^{j^*} a_{(i)}$ and thus $j^* a_{(j^*+1)} \geqslant \sum_{i=1}^{j^*+1} a_{(i)}$. This implies that (12) does not hold for $j^* + 1$ instead of j^*. Also, for $k \geqslant 2$,

$$(j^* + k - 1)a_{(j^*+k)} \geqslant j^* a_{(j^*+1)} + a_{(j^*+2)} + \ldots + a_{(j^*+k)} \geqslant \sum_{i=1}^{j^*+k} a_{(i)}.$$

So (12) does not hold for any $j^* + k$, $k \geqslant 2$.

(ii) Clearly, if both inequalities in (12) are strict for j^*, then T is continuous at $(a_1, \ldots, a_n)$. If the second inequality is an equality, then it is sufficient to prove that the value of T does not change if we replace j^* by $j^* + 1$, i.e., that

$$(j^* - 1)/\sum_{i=1}^{j^*} a_{(i)} = j^*/\sum_{i=1}^{j^*+1} a_{(i)}. \tag{13}$$

To show this, write $T(j^*) = (j^* - 1)/\sum_{i=1}^{j^*} a_{(i)}$ and note that

$$\begin{aligned} \frac{1}{j^*}\sum_{i=1}^{j^*+1} a_{(i)} &= \frac{(j^*-1)\frac{1}{j^*-1}\sum_{i=1}^{j^*} a_{(i)} + a_{(j^*+1)}}{j^*} \\ &= \frac{(j^*-1)\frac{1}{T(j^*)} + \frac{1}{T(j^*)}}{j^*} = \frac{1}{T(j^*)}, \end{aligned}$$

implying (13). □

Proof of Lemma 3.4. For convenience we drop s from the notations in this proof. We write $a_i = \lambda_i/u_i$ for every i and without loss of generality assume that $a_1 \leqslant \ldots \leqslant a_n$. Let $I = \{i \in N \mid \sigma_i > 0\}$. The proof proceeds by two claims.

Claim 1: Let $j \in I$ and $k < j$. Then $k \in I$.

To prove this claim, note that $\lambda_j = y_j u_j/T^2 < T u_j/T^2 = u_j/T$, so $\lambda_j/u_j < 1/T$. Hence $y_k/T_k^2 \leqslant \lambda_k/u_k = a_k \leqslant a_j = \lambda_j/u_j < 1/T$. This implies $y_k < T$ and, thus, $\sigma_k > 0$, i.e., $k \in I$.

Claim 1 implies that $I = \{1, \ldots, j\}$ for some $j \in N$.

Claim 2: Let $I = \{1, \ldots, j\}$. Then

$$a_j < \frac{1}{j-1} \sum_{i=1}^{j} a_i \leqslant a_{j+1}. \tag{14}$$

To prove this, first note that $\sigma_i = T - \lambda_i T^2/u_i = T - a_i T^2$ for all $i \in I$. This implies $T = \sum_{i \in I} \sigma_i = jT - \sum_{i \in I} a_i T^2$, and thus $1/T = [1/(j-1)] \sum_{i=1}^{j} a_i$. Then

$$a_j = \frac{y_j}{T^2} < \frac{T}{T^2} = \frac{1}{T} = \frac{1}{j-1} \sum_{i=1}^{j} a_i,$$

and

$$a_{j+1} \geqslant \frac{y_{j+1}}{T^2} = \frac{T}{T^2} = \frac{1}{T} = \frac{1}{j-1} \sum_{i=1}^{j} a_i.$$

This completes the proof of Claim 2. The first part of the lemma now follows from part (i) of Lemma A.1. By part (ii) of Lemma A.1, $T(s)$ depends continuously on $(\lambda_i, u_i(s))_{i \in N}$, and therefore $\sigma(s)$ depends continuously on $(\lambda_i, u_i(s))_{i \in N}$ as well. □

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