The core of multi-sided assignment games: axiomatization and extensions

Marina Núñez

Universitat de Barcelona

Corvinus University Budapest, December 2013

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Coalitional TU games

A cooperative game with transferable utility is (N, v), where

- $N = \{1, 2, \ldots, n\}$ is the set of players and
- $\begin{array}{rcl} v : 2^N & \longrightarrow & \mathbb{R} \\ S & \mapsto & v(S) \end{array}$ is the characteristic function.

An imputation is a payoff vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$ that is

- Efficient: $\sum_{i \in N} x_i = v(N)$
- Individually rational: $x_i \geq v(i)$ for all $i \in N$.

Let $I(v)$ be the set of imputations of (N, v).

The core is the set of imputations that are coalitionally rational:

$$C(v) = \{x \in I(v) \mid x(S) \geq v(S), \text{ for all } S \subset N\}.$$

A game may have an empty core.

The dominance relation and the core

Let it be (N, v) and $x, y \in I^*(v)$:

- y dominates x via coalition $S \neq \emptyset$ $(y\, dom_S^v x) \Leftrightarrow x_i < y_i$ for all $i \in S$ and $\sum_{i \in S} y_i \leq v(S)$.
- y dominates x $(y\, dom^v x)$ if $y\, dom_S^v x$ for some $S \subseteq N$.

Definition (Gillies, 1959)

The core $C(v)$ of (N, v) is the set of preimputations undominated by another preimputation.

- If $C(v) \neq \emptyset$, then it coincides with the set of imputations undominated by another imputation.

The stable sets or von Neumann-Morgenstern solutions

Definition (von Neumann and Morgenstern, 1944)

Given (N, v), a subset of imputations $V \subseteq I(v)$ is a stable set if

1. two imputations $x, y \in V$ do not dominate one another (internal stability) and
2. any $y \in I(v) \setminus V$ is dominated by some $x \in V$ (external stability).

- The core is always included in any stable set.
- There are games with no stable set (Lucas, 1968, 1969).

The assignment market

- A two-sided assignment market $\gamma = (M, M', A, p, q)$ is defined by a finite set of buyers M, a finite set of sellers M', a matrix $A = (a_{ij})_{(i,j)\in M\times M'}$, and two vectors of reservation values, $p \in \mathbb{R}^M$, and $q \in \mathbb{R}^{M'}$.
- A good is traded in indivisible units (side-payments allowed).
- Each buyer in $M = \{1, 2, \ldots, m\}$ demands one unit and each seller in $M' = \{1, 2, \ldots, m'\}$ supplies one unit.
- Buyer i and seller j make a join profit of a_{ij} if they trade.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m'} \\ a_{21} & a_{22} & \ldots & a_{2m'} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \cdots & a_{mm'} \end{pmatrix}$$

- p_i is the reservation value of buyer $i \in M$ and q_j the reservation value of seller $j \in M'$.
- Neither a_{ij} nor p_i or q_j are constrained to be non-negative.

The assignment market

- A matching μ between $S \subseteq M$ and $T \subseteq M'$ is a bijection from a subset of S to a subset of T. $Dom(\mu) \subseteq S$ and $Im(\mu) \subseteq T$ denote, respectively, the corresponding domain and image.
- Given an assignment market $\gamma = (M, M', A, p, q)$, for all $S \subseteq M$, $T \subseteq M'$ and $\mu \in \mathcal{M}(S,T)$ we write

$$v(S,T;\mu) = \sum_{(i,j)\in\mu} a_{ij} + \sum_{i\in S\setminus Dom(\mu)} p_i + \sum_{j\in T\setminus Im(\mu)} q_j$$

- A matching $\mu \in \mathcal{M}(M,M')$ is optimal if for all $\mu' \in \mathcal{M}(M,M')$, $v(M,M';\mu) \geq v(M,M';\mu')$.

The assignment game

- The coalitional game associated to the assignment market $\gamma = (M, M', A, p, q)$ is defined by $(M \cup M', w_\gamma)$, where, for all $S \subseteq M$ and $T \subseteq M'$,

$$w_\gamma(S \cup T) = max\,\{v(S,T;\mu) \mid \mu \in \mathcal{M}(S,T)\}\,.$$

✓ This assignment game is a **generalization of the assignment game of Shapley and Shubik (1972)** and has been introduced by Owen (1992) and also used by Toda (2003, 2005) to axiomatize the core.
✓ This generalized assignment game is strategically equivalent to the Shapley-Shubik assignment game, and thus the core has the same structure.

- We denote by Γ_{AG} the set of all assignment markets.

The core of the assignment game

From Shapley and Shubik (1972) and because of strategic equivalence:

- Assignment games have a non-empty core.
- Given an optimal matching μ, $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ is in the core iff
  $u_i \geq p_i$, for all $i \in M$; $v_j \geq q_j$, for all $j \in M'$,
  $u_i + v_j \geq a_{ij}$ for all $(i, j) \in M \times M'$,
  $u_i + v_j = a_{ij}$ for all $(i, j) \in \mu$,
  $u_i = p_i$ and $v_j = q_j$ if $i \in M \setminus Dom(\mu)$ and $j \in M' \setminus Im(\mu)$.
- Notice that in the core of the assignment game third-party payments are excluded.

Solutions on the domain of assignment games

The next two definitions follow Toda (2005)

Definition

Let $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$. A payoff vector $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ is *feasible* if there exists $\mu \in \mathcal{M}(M, M')$ such that

(i) $u_i = p_i$ for all $i \in M \setminus Dom(\mu)$,

(i) $v_j = q_j$ for all $j \in M' \setminus Im(\mu)$, and

(iii) $u_i + v_j = a_{ij}$ for all $(i, j) \in \mu$.

✓ In the above definition, μ is said to be *compatible* with (u, v).

✓ A *compatible matching* need not be an optimal matching

Definition

A *solution* on Γ_{AG} is a correspondence σ that associates a nonempty subset of feasible payoff vectors with each $\gamma \in \Gamma_{\mathrm{AG}}$.

Derived game (Owen, 1992)

Definition

Let $\gamma = (M, M', A, p, q)$ be an assignment market, $\emptyset \neq T \subset M \cup M'$, and $x = (u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$. The **derived assignment market** relative to T at x is $\gamma^{T,x} = \left(T \cap M, T \cap M', A^T, p^{T,x}, q^{T,x}\right)$, where $A^T = A_{|(T\cap M)\times(T\cap M')}$ and

$$
\begin{aligned}
p_i^{T,x} &= \max\left\{p_i, \max_{j\in M'\setminus T}\{a_{ij} - v_j\}\right\}, \text{ for all } i \in T \cap M,\\
q_j^{T,x} &= \max\left\{q_j, \max_{i\in M\setminus T}\{a_{ij} - u_i\}\right\}, \text{ for all } j \in T \cap M'.
\end{aligned}
$$

The **derived assignment game** relative to T at x is the coalitional game associated to the derived assignment market $\gamma^{T,x}$, that is $(T, w_{\gamma^{T,x}})$.

Derived consistency

Definition

A solution σ on Γ_{AG} satisfies

- **derived consistency** if for all $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$, all $\emptyset \neq T \subset M \cup M'$ and all $x \in \sigma(\gamma)$, then $x_{|T} \in \sigma\left(\gamma^{T,x}\right)$.

- The derived game of an assignment game at a given core allocation coincides with the superadditive cover of the Davis and Maschler reduced game at this same core allocation (Owen, 1992).

Some known axiomatizations for two-sided assignment games

- Toda (2003) provides an axiomatization of the core on Γ_{AG} following Peleg (1986) by means of
  - Pareto optimality
  - individual rationality
  - (derived) consistency
  - super-additivity
- Toda (2005) axiomatizes the core on Γ_{AG} by means of
  - Pareto optimality
  - (projected) consistency
  - pairwise monotonicity
  - individual monotonicity (or population monotonicity)
- Van den Brink and Pinter (2012) characterize Shapley value
  - Submarket efficiency
  - Valuation fairness
- Llerena et al. (2012) characterize the nucleolus on Γ_{AG} by
  - (derived) consistency
  - symmetry of the maximum complaint of the two sides

A core extension: stable sets

- For the assignment game, dominance of imputations is always done via mixed-pair coalitions: $\{i, j\}$ with $i \in M$ and $j \in M'$.

Definition (Solymosi and Raghavan, 2001)

An assignment game $(M \cup M', w_A)$ with as many buyers as sellers has a dominant diagonal if for any optimal matching μ and all $k \in M$,

$$a_{k\mu(k)} \geq \max\{a_{kj}, a_{i\mu(k)}\}, \text{ for all } j \in M' \text{ and } i \in M \,.$$

Theorem (Solymosi and Raghavan, 2001)

An assignment market $(M \cup M', w_A)$ with as many buyers as sellers has a stable core iff it has a dominant diagonal.

The compatible subgames

Definition

Let $(M \cup M', w_A)$ be an assignment game.
Let μ be an optimal matching, $I \subseteq M$ and $J \subseteq M'$.
The subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is μ-compatible if

$$w_A((M\setminus I)\cup(M'\setminus J)) + \sum_{i\in I\cap\mu^{-1}(M')} a_{i\mu(i)} + \sum_{j\in J\cap\mu(I)} a_{\mu^{-1}(j)j} = w_A(M\cup M')$$

Example

	1'	2'	3'
1	5	**8**	2
2	7	9	**6**
3	**2**	3	0

$I = \{2\},\ J = \emptyset$
$I = \{2,3\},\ J = \emptyset$
$I = \{2\},\ J = \{1'\}$

$I = \emptyset,\ J = \{1'\}$
$I = \emptyset\ \ J = \{1', 2'\}$

The stable set in the μ-principal section

Theorem

Let $(M \cup M', w_A)$ and μ an optimal matching. The union of the extended cores of all the μ-compatible subgames, $V^\mu(w_A)$, is a stable set:

$$V^\mu(w_A) = \bigcup_{(I,J)\in \mathcal{C}_A^\mu} \hat{C}(w_{A_{-I\cup J}}),$$

where

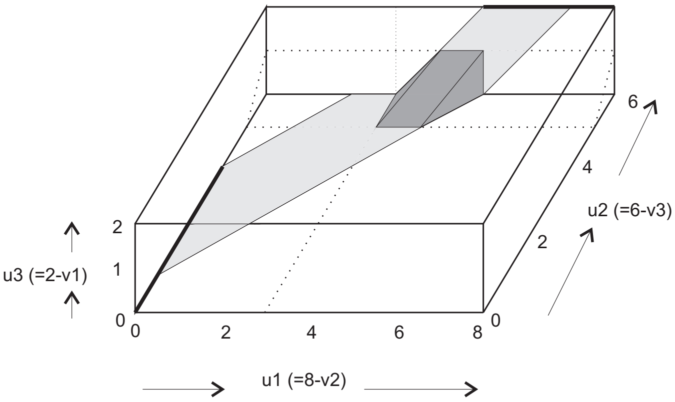
$$\mathcal{C}_A^\mu = \{(I,J) \in 2^M \times 2^{M'} \mid ((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}}) \ \mu\text{-compatible}\}.$$

Example 3: Shapley and Shubik, 1972

	1'	2'	3'
1	5	**8**	2
2	7	9	**6**
3	**2**	3	0

$I = \{2\},\ J = \emptyset$
$I = \{2,3\}\ ,\ J = \emptyset$
$I = \{2\}\ ,\ J = \{1'\}$

$I = \emptyset\ ,\ J = \{1'\}$
$I = \emptyset\ \ J = \{1', 2'\}$



Multi-sided assignment markets

An **m-sided assignment market (m-SAM)**, $\gamma = (N_1, \ldots, N_m; A, \pi)$, consists of:

- $m \geq 2$ **different finite sets** or **sectors** $N_1 \subseteq \mathcal{U}_1, \ldots, N_m \subseteq \mathcal{U}_m$ of cardinality $|N_i| = n_i$ where $n_i \in \mathbb{N} \cup \{0\}$, for $i \in \{1, \ldots, m\}$, with $N = \bigcup_{r=1}^{m} N_r$ non-empty,
- an **m-dimensional matrix** $A = (a_E)_{E \in \prod_{r=1}^{m} N_r}$, and
- a **vector of reservation values** $\pi \in \mathbb{R}^N$, where π_i stands for the reservation value of player $i \in N$.

✓ We name any m-tuple of agents $E = (i_1, \ldots, i_m) \in \prod_{r=1}^{m} N_r$ an **essential coalition**.

✓ Neither a_E nor π_i **are constrained to be non-negative**.

Multi-sided assignment markets

- A **matching** μ among $S_1 \subseteq N_1, \ldots, S_m \subseteq N_m$ is either $\mu = \emptyset$ or a set of pairwise disjoint essential coalitions, that is, $\mu = \{E^r\}_{r=1}^t$, with $1 \leq t \leq \min\{|S_1|, \ldots, |S_m|\}$ and $E^i \cap E^j = \emptyset$ for $i, j \in \{1, \ldots, t\}$, $i \neq j$. We denote by $\mathcal{M}(S_1, \ldots, S_m)$ the set of all matchings among $S_1, \ldots, S_m$.
- The **support of a matching** $\mu \in \mathcal{M}(S_1, \ldots, S_m)$ is defined by $supp\,(\mu) = \{k \in N \mid \text{there is } E \in \mu \text{ and } k \in E\}$.
- Given $\mu \in \mathcal{M}(N_1, \ldots, N_m)$ and $\emptyset \neq T \subseteq N$, $T \neq N$, we define $\mu_{|T} = \{E \in \mu \,|\, E \in \prod_{r=1}^m (N_r \cap T)\}$.
- Given $\gamma = (N_1, \ldots, N_m; A, \pi)$, for all $S_1 \subseteq N_1, \ldots, S_m \subseteq N_m$ and all $\mu \in \mathcal{M}(S_1, \ldots, S_m)$, we write

$$v(S_1, \ldots, S_m; \mu) = \sum_{E \in \mu} a_E + \sum_{\substack{i \in \bigcup_{r=1}^m S_r \\ i \notin supp\,(\mu)}} \pi_i .$$

Multi-sided assignment markets

- A **matching** $\mu \in \mathcal{M}(N_1, \ldots, N_m)$ is **optimal** for $\gamma = (N_1, \ldots, N_m; A, \pi)$, if for all $\mu' \in \mathcal{M}(N_1, \ldots, N_m)$ it holds $v(N_1, \ldots, N_m; \mu) \geq v(N_1, \ldots, N_m; \mu')$.
- The set of optimal matchings for γ is denoted by $\mathcal{M}^*_\gamma(N_1, \ldots, N_m)$.
- With any $\gamma = (N_1, \ldots, N_m; A, \pi)$, we associate a **cooperative game**, denoted by (N, w_γ), with player set $N = \bigcup_{r=1}^m N_r$ and characteristic function

$$w_\gamma(S_1 \cup \ldots \cup S_m) = \max_{\mu \in \mathcal{M}(S_1, \ldots, S_m)} \{v(S_1, \ldots, S_m; \mu)\},$$

for all $S_1 \subseteq N_1, \ldots, S_m \subseteq N_m$.

Multi-sided assignment markets

✓ This m-sided assignment game, that allows for agents' reservation values, is a **generalization of the m-sided assignment game of Quint (1991)**, where the reservation values are null and the assignment matrix is non-negative.

✓ Every assignment game in Γ_{m-SAM} is **strategically equivalent** to an m-sided assignment game in the sense of Quint (1991). As a consequence, **Quint's results on the core of the m-sided assignment game with non-negative matrix and null reservation values extend to** Γ_{m-SAM}.

Multi-sided assignment markets: The core

Fact

The **CORE** *of the m-sided assignment market is formed by those* **efficient** *payoff vectors $u \in \mathbb{R}^N$ satisfying* **coalitional rationality for essential and one-player coalitions***:*

$$C(\gamma) = \left\{ u \in \mathbb{R}^N \;\middle|\; \begin{array}{l} u(N) = w_\gamma(N), \\ \sum_{i \in E} u_i \geq a_E, \forall E \in \prod_{r=1}^m N_r, \\ u_i \geq \pi_i, \forall i \in N \end{array} \right\}.$$

If μ is an **optimal matching** of $\gamma \in \Gamma_{m-SAM}$, any core allocation $u \in \mathbb{R}^N$ satisfies

$$\sum_{i \in E} u_i = a_E \text{ for all } E \in \mu, \text{ and}$$
$$u_i = \pi_i \text{ for all } i \in N \setminus supp(\mu).$$

An example with an empty core

k_1	j_1	j_2
i_1	**6**	8
i_2	2	1

k_2	j_1	j_2
i_1	7	1
i_2	1	**4**

- The optimal matching is $\mu = \{(1,1,1),(2,2,2)\}$.
- A core element $(x_1, x_2; y_1, y_2; z_1, z_2)$ should satisfy

$$\begin{aligned} x_1 + y_1 + z_1 &= 6, & x_2 + y_2 + z_2 &= 4, \\ x_1 + y_2 + z_1 &\geq 8, & x_1 + y_1 + z_2 &\geq 7, \\ x_2 + y_1 + z_1 &\geq 2, & x_1 + y_2 + z_2 &\geq 1, \\ x_2 + y_2 + z_1 &\geq 1, & x_2 + y_1 + z_2 &\geq 1. \end{aligned}$$

- Adding up we get a contradiction

$$2(x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \geq 17$$

and hence there are no core elements.

Some references on multi-sided assignment games

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- Quint, *The core of an m-sided assignment game.* GEB, 1991.
- Lucas, *Core theory for multiple sided assignment games.* Duke Math J., 1995.
- Stuart, *The supplier-firm-buyer game and its M-sided generalization.* MSS, 1997.
- Sherstyuk, *Multisided matching games with complementarities.* IJGT, 1999.
- Tejada and Rafels, *Symmetrically multilateral-bargained allocations in multi-sided assignment markets.* IJGT, 2010.
- Tejada, *Multi-sided Böhm-Bawerk assignment markets: the core.* 2010.
- Tejada and Núñez, *The nucleolus and the core center of multi-sided Böhm-Bawerk assignment markets.* 2010.

Solution

Definition (Toda, 2005)

Let $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma_{m-SAM}$. A payoff vector $u \in \mathbb{R}^N$ is **feasible** for γ if there exists a matching $\mu \in \mathcal{M}(N_1, \ldots, N_m)$ such that:

1. $\sum_{i \in E} u_i = a_E$ for all $E \in \mu$, and
2. $u_i = \pi_i$ for all $i \in N \setminus supp\,(\mu)$.

✓ In the above definition, μ is said to be **compatible** with u.

✓ A **matching** that is **compatible with a feasible payoff vector needs not be an optimal matching**.

✓ Taking $\mu = \emptyset$, we have that $u = \pi$ is a **feasible payoff vector**.

Definition

A **solution** on $\Gamma \subseteq \Gamma_{m-SAM}$ is a correspondence σ that assigns a subset of feasible payoff vectors to each $\gamma \in \Gamma$.

Axioms

- **Non-emptiness (NE)**. A solution σ on Γ satisfies **NE** if for all $\gamma \in \Gamma$, $\sigma(\gamma) \neq \emptyset$.
- **Singleness best (SB)**. A solution σ on Γ satisfies **SB** if for all $\gamma \in \Gamma$ such that $\mu = \emptyset$ is an optimal matching, it holds $\pi \in \sigma(\gamma)$.

✓ **Singleness best** simply says that if remaining unmatched is optimal for every player, then the vector of reservation values should be an outcome of the solution.

Axioms

- **Individual anti-monotonicity (IAM)**. A solution σ on Γ satisfies **IAM** if for all $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma$, all $\gamma' = (N_1, \ldots, N_m; A, \pi') \in \Gamma$ and all $u \in \sigma(\gamma)$, if $\pi \geq_\mu \pi'$ for some matching μ compatible with u in γ, then $u \in \sigma(\gamma')$.

✓ $\pi \geq_\mu \pi'$ means that

1. $\pi_i = \pi'_i$ if $i \notin supp\,(\mu)$, and
2. $\pi_i \geq \pi'_i$ if $i \in supp\,(\mu)$,.

✓ Individual anti-monotonicity is a weaker version of **anti-monotonicity** introduced by Keiding (1986) and also used by Toda (2003).

Axioms: Derived consistency

Definition

Let $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma_{m-SAM}$, $\emptyset \neq T \subseteq N$, $T \neq N$, and $u \in \mathbb{R}^N$. The **derived m-sided assignment market** of γ relative to T at u is $\gamma^{T,u} = (N_1 \cap T, \ldots, N_m \cap T; A^T, \pi^{T,u})$, where $A^T = (a_E)_{E \in \prod_{r=1}^m (N_r \cap T)}$ and $\pi^{T,u} \in \mathbb{R}^T$ is the vector of reservation values defined as follows:

$$\pi_i^{T,u} = \max\left\{\pi_i, \max_{\substack{E \in \prod_{r=1}^m N_r,\, i \in E \\ E \cap (N \setminus T) \neq \emptyset}} \left\{a_E - \sum_{j \in E_{-\{i\}}} u_j\right\}\right\}, \forall i \in N$$

Axioms: Derived consistency

- **Derived consistency (D-CONS)**. A solution σ on Γ satisfies **D-CONS** if for all $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma$, all $\emptyset \neq T \subseteq N$, $T \neq N$, and all $u \in \sigma(\gamma)$, then $\gamma^{T,u} \in \Gamma$ and $u_{|T} \in \sigma(\gamma^{T,u})$.

Proposition

On the domain of m-sided assignment games Γ_{m-SAM}, the core satisfies derived consistency.

Axioms: Projection consistency

Definition

Let $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma_{m-SAM}$, $\emptyset \neq T \subseteq N$, $T \neq N$ and $u \in \mathbb{R}^N$ a feasible payoff vector compatible with a matching $\mu \in \mathcal{M}(N_1, \ldots, N_m)$. An m-sided assignment market $\gamma_p^{T,u} = (N_1 \cap T, \ldots N_m \cap T; A^T, \pi^T)$ is said to be the **projected assignment market** of γ relative to T at u and μ if the following conditions are satisfied:

1. If $E \in \mu$ and $E \cap T \neq \emptyset$, then $E \subseteq T$.
2. $A^T = (a_E)_{E \in \prod_{r=1}^m (N_r \cap T)}$.
3. For all $i \in T$, $\pi_i^T = \pi_i$.

✓ The **projected market** relative to T at u and μ is just the submarket associated to coalition T (under the requirement that T must not separate members of any essential coalition in μ).

Axioms: Projection consistency

- **Projection consistency (P-CONS)**A solution σ on Γ satisfies **P-CONS** if for all $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma$, all $\emptyset \neq T \subseteq N$, $T \neq N$, and all $u \in \sigma(\gamma)$, then $\gamma_p^{T,u} \in \Gamma$ and $u_{|T} \in \sigma\left(\gamma_p^{T,u}\right)$.

✓ On the domain of two sided assignment games, Toda (2005) uses projection consistency to characterize the core.

Proposition

On the domain of balanced m-sided assignment games Γ_c, the core satisfies projection consistency.

On the domain of balanced m-sided assignment games

Proposition

Let σ be a solution on $\Gamma \subseteq \Gamma_{m-SAM}$ satisfying derived consistency. Then, for all $\gamma \in \Gamma$, $\sigma(\gamma) \subseteq C(\gamma)$.

Proposition

Let σ be a solution on Γ_c satisfying non-emptiness and projection consistency. If for all $\gamma \in \Gamma_c$, $\sigma(\gamma) \subseteq C(\gamma)$ then, for all $\gamma \in \Gamma_c$, $\sigma(\gamma) = C(\gamma)$.

Theorem

On the domain of balanced m-sided assignment games Γ_c, the core is the only solution satisfying non-emptiness, derived consistency and projection consistency.

On the whole domain of m-sided assignment games

Proposition

Let σ be a solution on Γ_{m-SAM} satisfying singleness best and individual anti-monotonicity. Then, for all $\gamma \in \Gamma_{m-SAM}$, $C(\gamma) \subseteq \sigma(\gamma)$.

From the above proposition, and taking into account that derived consistency implies core selection, we obtain a characterization of the core on the domain of all m-sided assignment markets:

Theorem

On the domain of all m-sided assignment games Γ_{m-SAM}, the core is the only solution satisfying singleness best, individual anti-monotonicity and derived consistency.

On the domain of two-sided assignment games

Corollary

On the domain of two-sided assignment games, the core is the only non-empty solution satisfying derived consistency and projection consistency.

Corollary

On the domain of two-sided assignment games, the core is the only solution satisfying singleness best, individual anti-monotonicity and derived consistency.

Independence of the axioms

Example

For all $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma_{m-SAM}$, let us consider

$$\sigma_1(\gamma) = \emptyset.$$

✓ σ_1 satisfies **D-CONS**, **P-CONS** and **IAM**, but not **NE** and **SB** on both Γ_{m-SAM} and Γ_c.

Example

For all $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma_{m-SAM}$, let us consider

$$\sigma_2(\gamma) = \left\{ u \in \mathbb{R}^N \;\middle|\; \begin{array}{l} u \text{ feasible for } \gamma, \\ u_i \geq \pi_i, \forall i \in N, \\ u(N) = w_\gamma(N) \end{array} \right\}.$$

✓ σ_2 satisfies **P-CONS**, **SB**, **IAM**, **NE** but not **D-CONS** on both Γ_{m-SAM} and Γ_c.

Independence of the axioms

Example

For all $\gamma = (N_1, \ldots, N_m; A, \pi) \in \Gamma_{m-SAM}$, let us consider

$$\sigma_3(\gamma) := \begin{cases} \emptyset, & \text{if } \gamma \notin \Gamma_c \\ SMB(\gamma), & \text{if } \gamma \in \Gamma_c. \end{cases},$$

where *SMB* is the set of **symmetrically multilateral-bargained allocations** (Tejada and Rafels, 2010).

✓ σ_3 satisfies **SB**. On the domain of Γ_c, σ_3 satisfies **D-CONS** and **NE**. Trivially σ_3 satisfies **D-CONS** on Γ_{m-SAM}, but not **P-CONS** on Γ_c and **IAM** on Γ_{m-SAM}.

✓ Thus, σ_1, σ_2 and σ_3 show the independence of the axioms in the above characterization results.

Extensions of the core (with Ata Atay)

Since the core of an m-sided assignment game may be empty, it is interesting to analyze non-empty core extensions, in particular stable sets. First we generalize the dominant diagonal condition.

Definition

Let $\gamma = (N_1, \ldots, N_m; A) \in \Gamma_{m-SAM}$, and μ an optimal matching. γ has a dominant diagonal iff

$$a_E \geq a'_E \text{ for all } E \in \mu \text{ and all } E' \cap E \neq \emptyset.$$

Fact

If $C(\gamma)$ is a stable set, then γ has a dominant diagonal.

Fact

If γ is $2 \times 2 \times 2$, then
$C(\gamma) \neq \emptyset$ iff γ has a dominant diagonal.

μ-compatible subgames

Definition

Given $\gamma = (N_1, \ldots, N_m, A)$ is a m-sided assignment market and μ an optimal matching. Let $I = \bigcup_{k=1}^{m} I_k$. Then $\gamma_{-I} = (N_1 \setminus I_1, \ldots, N_m \setminus I_m, A_{-I})$ is a μ-compatible subgame iff

$$w_\gamma\left(\bigcup_{k=1}^{m}(N_i \setminus I_k)\right) + \sum_{k \in I \cap supp(\mu)} a_{E^k} = w_\gamma(N)$$

where $k \in E^k \in \mu$.

Take now the union of the cores of all μ-compatible subgames of γ:

$$V^\mu(\gamma) = \bigcup_{I \in \mathcal{C}^\mu(\gamma)} \hat{C}(\gamma_{-I})$$

where $\mathcal{C}^\mu(\gamma) = \{R \subseteq N \mid w_{-R} \text{ is a } \mu\text{-compatible subgame of } w_\gamma\}$.

An internally-stable extension of the core

- For all γ and μ, $V^{\mu}(\gamma)$ is non-empty, contains the allocations $a^k \in \mathbb{R}^N$ where

$$a_j^k = \begin{cases} a_E & \text{if } j \in N^k, j \in E \in \mu \\ 0 & \text{otherwise} \end{cases}$$

- $V^{\mu}(\gamma)$ is internally stable.
- If γ has a dominant diagonal, then $V^{\mu}(\gamma) = C(\gamma)$.

Example

But $V^{\mu}(\gamma)$ is not in general a stable set

k_1	j_1	j_2
i_1	**3**	1
i_2	2	5

k_2	j_1	j_2
i_1	1	4
i_2	5	**4**

The core is empty and $V^{\mu}(\gamma) = \{a^1, a^2, a^3\}$.

The core with respect to the μ-principal section

Given a m-sided assignment game $\gamma = (N_1, \ldots, N_m; A)$ and an optimal matching μ, consider the set of allocations that only allow for side payments between agents optimally matched by μ:

$$B^\mu(\gamma) = \left\{ u \in \mathbb{R}_+^N \mid \begin{array}{ll} u(E) = a_E & \text{for all } E \in \mu \\ u_i = 0 & \text{for all } i \notin supp(\mu) \end{array} \right\}$$

This set is named the μ-principal section of γ.

Fact

$V^\mu(\gamma)$ is the set of undominated allocations w.r.t. the μ-principal section.

That is, the set $V^\mu(\gamma)$ is the core with respect to the μ-principal section.

THANK YOU